THE DEVELOPMENT OF THE EQUAL TEMPERAMENT SCALE

EVOLUTION OR RADICAL CHANGE?

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ABSTRACT

This thesis examines the developments that preceded the acceptance of the equal temperament scale as a standard tuning for fixed pitch instruments. The question to be answered is whether the change to equal temperament scale was part of a natural progression over time, or a discovery that made radical changes to the accepted tuning of the day. The first section of the thesis explains and gives examples of the connections between mathematics and pitch. Next, the history of tuning from prehistoric times to eighteenth century Europe is outlined. The final sections explain and interpret the analysis done for this study.

Specifically answering the question for this thesis, equal temperament was a radical change in tuning. The analysis also showed that every tuning could be described as a radical change from its predecessor because no trends were evident. Regardless, favoring a tonality system in the eighteenth century and keeping it as a standard for over two hundred years is a radical change, considering the amount of variability throughout the previous three centuries.

This study only provides evidence that the transition to equal temperament was not a natural one, but was driven by necessity. Future studies are needed by mathematicians and musicologists to assess the impact of equal temperament on the human interpretation of intervals and harmony. Having been conditioned to equal temperament for two centuries, it is questionable if a change to our current tuning method for fixed pitch instruments is desired.

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CHAPTER 1

INTRODUCTION

A debate exists in the music community about the current method in which fixed pitched instruments such as pianos are tuned. This debate dates back to the mid seventeenth century when equal temperament tuning was first invented. Some in the music community believe the equal tempered scale has lessened the effectiveness of harmony and that harmony has been compromised by the invention. Even the first Western European to publish it, Marin Mersenne (b. 1588), developed other tunings besides equal temperament.

On the other side of the debate, many have pointed out the great benefits of equal tempered tuning. Composers now write music that would have been impossible to enjoy without the tempered scale. Instruments have also advanced with more mechanical assistance available for wind and string performers. Most importantly, our society has (unknowingly) accepted the equal tempered scale as the norm. Switching back to any of the historic systems of tuning would be quite noticeable to most of the public and would force musicians and composers to alter their techniques.

This thesis does not argue for either side of the debate. Instead, it examines the developments that preceded the acceptance of the equal tempered scale as a standard tuning for western music. The question to be answered is not if the tuning is good or bad; the question is whether the Equal Temperament Scale is part of a natural progression over time, or a discovery that made radical changes to the accepted tuning of the day. Although this will not resolve any arguments, it may shed light on reasons for the debate's existence.

The first section of the thesis explains and gives examples of the connections between mathematics and pitch. The formulas and units of measure are also laid out to help explain the debate. The second section outlines the history from ancient China, when it is believed people first explored the relationship between harmony and mathematics, to eighteenth century Europe, when the equal temperament scale was first applied to Western music. The most popular tuning methods throughout this time period are described in detail. These methods are chosen for analysis in the third section.

The third section explains the research completed for this study. Data organization, methods of analysis, and results are outlined both in this section and in the attached charts, graphs, and tables. Important trends, correlations, and anomalies are explored. In addition to the visual aids, I have also assembled musical excerpts so the reader may hear tuning methods as they are explained and analyzed. Links to these excerpts are available at http://thesis.grenfellmusic.net.

In the last section, the research and analysis are used to answer the question posed by this thesis. This question is an important one because the equal temperament scale has changed our lives. A person, who has heard nothing but equal temperament, listening to the older tuning methods is similar to a modern English speaking person listening to "old" English. Although there is some familiarity, the words and phrases sound foreign. The older tunings also sound familiar to a modern listener, but sound out of tune because the relationships between pitches are not exactly the same as those contained in the music we have been exposed to since childhood.

It would be remarkable to discover that the modern English language came to be without a progression of small changes over time; evidence to the contrary can be found in literature published throughout the centuries. It would also be remarkable if the underlying structure of music underwent a sudden change as opposed to an evolutionary process. The first sections of this paper describe a conflict between the goals of those who tried to "perfect" the musical scale and the reality of nature and sound. This conflict could have been the impetus for sudden change. Sudden change could be the cause for debate.

CHAPTER 2

WHAT IS PITCH?

Those presenting the mathematics/music connection often use rhythm to make their case. This makes sense because one is introduced to number values for note durations when receiving musical training. Pitch is another example of mathematics in music that is less commonly discussed. Pitch is taught completely by rote. The theory behind what makes one pitch match another is not discussed, because knowing the mathematics will not cause one to suddenly sing in tune. Although impractical as a teaching device, the application of pitch in music is purely mathematical.

Sounds are actually disturbances in air pressure caused by a vibration. These disturbances are longitudinal waves as opposed to transversal waves. The ripples created by a pebble dropped in calm water are transversal waves because the motion of each particle is perpendicular to the travel of the wave. Sound waves are longitudinal because the air molecules are repeatedly moving away and then back towards the sound source. Therefore, the particles are moving parallel to the direction of the wave. Another distinction between sound waves and ripples in a pond is that the areas of compression in the air can be pictured as a series of spheres (three-dimensional) growing from the sound source as opposed to circles (two-dimensional) from the pebble in water. Although distinct differences exist between sound and the pebble analogy often used to describe it, sound waves are often analyzed using two-dimensional drawings more akin to transversal waves. This is acceptable because it is much easier to visualize and the sound sources themselves (such as a vibrating string) fall under the category of transversal waves.[i]

Most sounds encountered in every day life have waves with irregular frequencies. Producing a sound wave with a regular frequency creates a single pitch in music. The frequency is measured in Hertz (Hz), which is the number of cycles per second. For example, the pitch *A* above middle *C* on a piano is usually tuned to a frequency of 440 Hz. Graphing the wave produced by a pitch results in a sine wave if all harmonics are absent (harmonics will be discussed in the next section). The vertical axis of this graph represents the level of compression in the air and the horizontal axis represents time.[ii] This graph is identical in shape to a graph representing the physical motion of the sound source. For this interpretation, the vertical axis represents the distance (positive or negative) a point on the object is displaced throughout the object's vibration, with zero representing where this point would be if the object were at rest. Studying the behavior of a string is a good way to understand why the graph of a pitch forms a sine wave.

The center point of a vibrating string is affected by a force proportional to the distance it has traveled from its resting point. This can be represented by the equation:

$$F = -ky$$

F is the force due to string tension, y is the distance from a resting point, and k is a constant of proportionality. Knowing that force equals mass, m, times acceleration, a, and acceleration is the second derivative of the displacement, y, the following equation can also describe this force:

$$F = ma = my''$$

By substitution these equations arise:

$$my'' = -ky$$
$$y'' = \frac{-ky}{m}$$
$$y'' + \frac{ky}{m} = 0$$

Since this is a study of pitch, this representation of a musical sound does not include harmonics, which will be discussed in the next chapter. The functions that satisfy the conditions for y in the second order differential equation above are in the form:

$$y = A\cos\left(\sqrt{\tfrac{k}{m}}t\right) + B\sin\left(\sqrt{\tfrac{k}{m}}t\right)$$

A and B are constants that depend on initial conditions of the vibration. Using trigonometric identities and substitution, the function can be rewritten in the form:

$$y = c\sin\left(\sqrt{\tfrac{k}{m}}t + \phi\right) \qquad \text{with :} \qquad c = \sqrt{A^2 + B^2}, \quad \phi = -\tan^{-1}\left(\tfrac{A}{B}\right)$$

With the function in this form, it is obvious that the wave produced by a string behaves exactly the same as a sine wave.[iii] It can further be concluded that a sound wave of a fixed pitch also behaves as a sine wave.

The above equation can be manipulated to produce a sine wave that matches a specific frequency. If A above middle C is the desired pitch, the frequency should be 440 Hz. The period of a sine wave is 2π, so the desired frequency can be made by setting $\sqrt{\tfrac{k}{m}}$ equal to $440 * 2\pi$, producing the equation $y = c\sin(880\pi t + \phi)$.[iv] Amplitude (volume of the string) is represented by c and phase (shifting of the wave from left to right) is represented by ϕ. It can be seen that solving for $\sqrt{\tfrac{k}{m}} = 880\pi$ for any given k or *m* (i.e. tension or mass) will reveal many possible ratios of these two conditions needed to produce the pitch *A* 440 Hz with a string.

For the purposes of this thesis, amplitude and phase will be ignored. Although they can have an effect on the perception of tuning (e.g. if one note is louder than another, the resulting beats may be less distinguishable), the theory of tuning is not affected by either of these elements. If two strings both tuned to *A* 440 Hz are played and each have different phases, ϕ_1 & ϕ_2, and amplitudes, c_1 & c_2, their combined waves can be expressed as:

$$y = c_1 \sin(880\pi t + \phi_1) + c_2 \sin(880\pi t + \phi_2)$$

Using the trigonometric identities, y can be expressed the following ways:

$$y = c_1 \sin(880\pi t + \phi_1) + c_2 \sin(880\pi t + \phi_2)$$
$$y = c_1(\sin(880\pi t)\cos(\phi_1) + \cos(880\pi t)\sin(\phi_1)) + c_2(\sin(880\pi t)\cos(\phi_2) + \cos(880\pi t)\sin(\phi_2))$$
$$y = c_1 \sin(880\pi t)\cos(\phi_1) + c_1 \cos(880\pi t)\sin(\phi_1)) + c_2 \sin(880\pi t)\cos(\phi_2) + c_2 \cos(880\pi t)\sin(\phi_2)$$
$$y = \sin(880\pi t)[(c_1 \cos(\phi_1) + c_2 \cos(\phi_2)] + \cos(880\pi t)[(c_1 \sin(\phi_1) + c_2 \sin(\phi_2)]$$

Since $[(c_1 \cos(\phi_1) + c_2 \cos(\phi_2)]$ and $[(c_1 \sin(\phi_1) + c_2 \sin(\phi_2)]$ are both functions of constants, they will be set to new constants, m_1 and m_2 respectively. With these new constants the following can be ascertained:

$$y = m_1 \sin(880\pi t) + m_2 \cos(880\pi t)$$
$$\text{let} \quad R\sin(880\pi t) = m_1 \sin(880\pi t) + m_2 \cos(880\pi t)$$

$$R\sin(880\pi t + \alpha) = R[\sin(880\pi t)\cos\alpha + \cos(880\pi t)\sin\alpha]$$
$$R\sin(880\pi t + \alpha) = R\sin(880\pi t)\cos\alpha + R\cos(880\pi t)\sin\alpha$$
$$R\sin(880\pi t + \alpha) = (R\cos\alpha)\sin(880\pi t) + (R\sin\alpha)\cos(880\pi t)$$
$$m_1 \sin(880\pi t) + m_2 \cos(880\pi t) = (R\cos\alpha)\sin(880\pi t) + (R\sin\alpha)\cos(880\pi t)$$
$$m_1 = R\cos\alpha$$
$$m_2 = R\sin\alpha$$
$$m_1^2 + m_2^2 = R^2\cos^2\alpha + R^2\sin^2\alpha$$
$$m_1^2 + m_2^2 = R^2(\cos^2\alpha + \sin^2\alpha)$$

$$m_1^2 + m_2^2 = R^2$$
$$R = \pm\sqrt{m_1^2 + m_2^2}$$
$$\frac{m_2}{m_1} = \frac{R\sin\alpha}{R\cos\alpha} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha$$
$$\alpha = \tan^{-1}\left(\frac{m_2}{m_1}\right)$$
$$m_1\sin(880\pi t) + m_2\cos(880\pi t) = \sqrt{m_1^2 + m_2^2}\sin\left(880\pi t + \tan^{-1}\left(\frac{m_2}{m_1}\right)\right)$$
$$y = \sqrt{m_1^2 + m_2^2}\sin\left(880\pi t + \tan^{-1}\left(\frac{m_2}{m_1}\right)\right)$$

This new representation of y shows that the resulting sound has the same pitch (*A* 440 Hz). When musicians tune their instruments, they do so by listening for beats while comparing two sounds. These beats are created by their pattern falling in and out of phase due to the two wavelengths being slightly different. As the beats slow down and eventually vanish, the two wavelengths are made the same and the two instruments are tuned to the same pitch. Since the above equation shows that amplitude and phase have no effect on the resulting pitch of two identical frequencies, a musician would hear no beats for this scenario because there is no discrepancy in wavelength.

The tuning method described above only works for unisons and octaves (frequencies that have a ratio that is 2^n with $n \in \mathrm{Z}$). The next section will explore the natural properties of sound that influenced our choices in the relationships between different pitches.

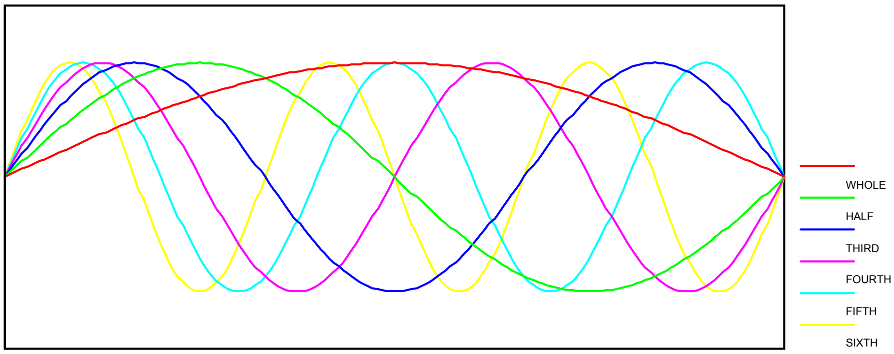
CHAPTER 3

WHAT ARE HARMONICS?

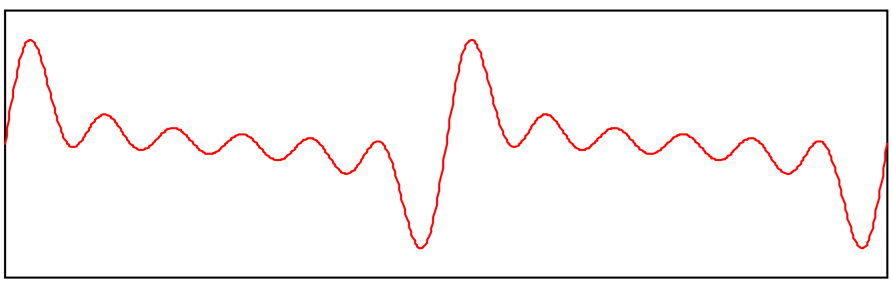
Both mathematicians and musicians use the word "harmonic". Many people do not know the connection between the musical and mathematical meaning of this word. However, the concept underlying both of the definitions is identical. From Ancient Greece through the Renaissance, the study of harmonics was not compartmentalized nearly as much as it is today. Music and mathematics were treated as one.

The harmonic series in mathematics is the pattern: $(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots)$. When an instrument creates a sound, this sound does not contain a single frequency. Many (theoretically infinite but in practice, fifteen at most) overtones are created above the fundamental pitch that contribute to the instrument's characteristic "voice" or timbre. Different instruments have different volume configurations for the set of overtones, but they are the same set of frequencies if each instrument is playing the same fundamental pitch. The frequencies of the fundamental tone and all possible overtones are known as harmonics. For a given frequency, m, the harmonics of a note tuned to a fundamental pitch m would be: $(m, 2m, 3m, 4m, \ldots)$.

Once again, a vibrating string is the best way to describe what creates these sounds. When a string vibrates, it does so as a whole. However, it also vibrates as if it were two strings each half the original's length, three strings of each a third the original's length, and so forth. The pivot points for these sub-vibrations are *nodes*.[v] The graph below represents a two dimensional cross-section of the first six vibrations in this series.



The sum of all these vibrations resembles the lines used to represent sounds in digital audio recording and editing software. The graph below represents the sum of all the waves in two cycles of the previous graph.



This naturally occurring pattern is the same as the harmonic series in mathematics. The equation for frequency of a string in relation to its physical characteristics is: $f = \frac{1}{2l}\sqrt{\frac{T}{m}}$, where *l* is length in centimeters, *T* is tension in dynes and *m* is density (grams per centimeter of string).[vi] Since the tension and density are theoretically constant, frequency is inversely proportional to the sub-lengths of a string. This explains the pattern for harmonic frequencies outlined above.

It should be noted that this representation of harmonics assumes a string is perfectly flexible, has a unison shape and density for its entire length, and has no restraints at its endpoints. Realistically, these conditions are never met; some imperfections are seen as positive characteristics of an instrument. Special tuning techniques are required to compensate for these imperfections (such as stretching octaves on a piano). These techniques are not in the scope of this paper because the tempered scale was developed as a mathematical solution for tuning one octave. Due to the relatively small range of one octave, ignoring these imperfections has little impact on the study of this tuning method.

As will be discussed in later sections, these harmonics became the foundation for rules of harmony and the development of the tempered scale. To understand our current tuning system, one has to understand the connection between harmonics and music.

CHAPTER 4

WHAT ROLE DO HARMONICS PLAY IN MUSIC?

Melody is an element of music shaped by harmonics. Melody is a series of pitches (frequencies) played one after another. Each has specific length and starting point (i.e. the series has a specific rhythm). Since an infinite number of frequencies exist, a limit must be present for the choice of pitches. In order to communicate, the performer must be “speaking a language” that is familiar to the listener. Harmonics influenced these choices, but they are still an infinite set. Also, the range covered in the beginning of the harmonic series would be taxing on a single human voice and therefore sound unnatural to the listener if used as a basis for melody.

The most important overtone in music is, not surprising, the first one. This harmonic creates the interval called an octave in music. Any two frequencies that have a ratio equal to $\frac{1}{2^n}$ with $n \in Z$ are called octaves of the same pitch. This is because their harmonic components are so similar, they sound as one when played in unison. Notes with this ratio are designated the same letter name in modern music notation. The other advantage of an octave is its range, which is comfortable enough to be sung. It is also wide enough to contain a variety of distinguishable pitches.

The number of frequencies within an octave is something that is less perfect in its development. We as humans have imitated the naturally occurring sounds that were pleasing and familiar. Unfortunately, the harmonics series that forged these sounds do not perfectly divide the octave range. Harmonic intervals higher than an octave can be moved down to the first octave by multiplying them by some power of two. Since the two notes will have a ratio of $\frac{1}{2^n}$ with $n \in Z$, the resulting pitch will be a different octave of the same note. For example, the seventh harmonic can be multiplied by four (two to the power of two) resulting in the ratio of four sevenths. Since this fraction falls between one and one-half, the resulting pitch is less than an octave above the fundamental frequency. Since the ratio between one-seventh and four sevenths is four (or $\frac{1}{2^{-2}}$), the frequency is a lower octave of the same pitch as the harmonic.

Repeating the process above creates a set of notes based on the natural harmonics within a reasonable range. Placing the more audible harmonics in the lower octave, people noticed that having twelve pitches seemed to divide the octave more evenly than adding more or less. In the table below, the simpler ratios that became part of our current chromatic scale (every pitch from one octave played from lowest to highest) are shown with their corresponding pitches. Many of these pitches cannot be found in the more audible harmonics with ratios consisting of numbers less than ten. The ratios (in decimal form) between each of these pitches are listed in the next row of the chart. The bottom row shows what the final ratio would be if twelve steps were made using each ratio. In this example, C is the fundamental pitch.

Dividing the Octave with Natural Harmonics

Pitch	C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C
Ratio	1	--	8/9	--	4/5	--	--	2/3	--	--	4/7	--	1/2
Ratio between pitches (decimal form)	.889 in 2 steps		.9 in 2 steps		.833 in 3 steps			.857 in 3 steps			.875 in 2 steps		
If ratio is extended to 12 steps	.494		.531		.481			.539			.449		

Although these more important harmonics do not perfectly fit into a scale divided into twelve equal parts, it can be seen why the tonal system scale evolved to contain this many pitches. The extension of these ratios by twelve come very close to the desired ending ratio of one half (or one octave). The mean of the extended ratios is (.499) with a standard deviation of (0.037). There are cultures that use quarter-tones (notes directly between two chromatics) in their music. The mean and standard variation for this scale (with twenty-four parts) is the same as the above example. However, quarter-tones are most often used as embellishments and are difficult for most listeners to distinguish.

While highlighting the strengths of a twelve-note chromatic scale, the calculations above also reveal the difficulty in tuning fixed pitched instruments. Since the ratios between tones derived by harmonics are not consistent, the scale is completely dependent upon the starting pitch. The most common scale in western music is called the major scale. It does not use all twelve notes of the chromatic. The chart below illustrates how a tuning method used before and during the early Renaissance would be used to create this scale. The “Just Intonation” scale is almost completely derived from the harmonics of a single pitch. The first row of values shows the tuning when applied to the tonic “C.” The second row shows what values would be needed to create another major scale using the

"D" from the first scale as its tonic. Notes left blank were not needed to play these scales.

Ratios Needed to Play in the Key of *C* and *D*

	C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C
C major	1:1	15:16	8:9	--	4:5	3:4	32:45	2:3	--	3:5	--	8:15	1:2
D major	--	128:135	8:9	--	64:81	--	32:45	2:3	--	16:27	--	24:45	--

Since four of these notes are different, it is impossible to tune a fixed pitch instrument for both of these major keys at once. To further illustrate this problem, the chart below uses decimal approximations for the ratios and adds the scales "E" through "B."

Ratios (decimal equivalents) Needed for Multiple Major Keys

	C	C#	D	D#	E	F	F#	G	G#	A	A#/Bb	B	C
C major	1.000	0.960	0.889	0.833	0.800	0.750	0.711	0.667	0.640	0.600	0.556	0.533	0.500
D major		0.948	0.889		0.790		0.711	0.667		0.593		0.533	
E major		0.960		0.853	0.800		0.711		0.640	0.600		0.533	
F major	1.000		0.900		0.800	0.750		0.667		0.600	0.563		0.500
G major	1.000		0.889		0.800		0.711	0.667		0.593		0.533	0.500
A major		0.960	0.900		0.800		0.720		0.640	0.600		0.533	
B major		0.948		0.853	0.800		0.711		0.640		0.569	0.533	

Every one of these scales has some variance from the ratios needed to play "C" Major. The "A#" or "Bb" has the most distinct ratios of all the pitches because it is the only one that is used as a sharp and a flat. Our equal tempered scale has conditioned us to think of them as one pitch with two names, but they are actually two separate pitches.

Adding the remaining major scales would create even more problems with more sharps and flats sharing the same fixed pitches.

One can guess the most commonly used keys and harmonies for a given period by looking at the imperfections of the prominent fixed pitch tuning methods of the time. Those scales with the least imperfections are likely the scales most used. In the next section, the most important tuning methods leading to and including Equal Temperament will be put in a historical context and explained. These are also the same tuning methods analyzed for this research.

CHAPTER 5

EARLY MUSICAL DEVELOPMENTS

Assumptions about the origins of music have recently been debunked by archeological finds and research. Many discoveries are directly related to the major scale. Those outlined in this section are examples of humans working to perfect an instrument's capability of playing the major scale, and are dated well before Ancient Greece. It is becoming more and more obvious that the ancient Greeks had many potential resources at their disposal when they made their contributions to music theory.

The oldest discovery is from China. In the Henan province, the remains of a Neolithic village were uncovered. Many artifacts found at this site were very advanced for their date. Among these artifacts, over thirty flutes made from the bone of cranes were between seven and nine thousand years old. Surprisingly, six of these flutes were in playable condition, and researchers recorded one of them playing a Chinese folk song called "Little Cabbage".[vii]

Comparing the pitches of each flute to their carbon-dated age has shown evidence of a progression to a seven-note scale. The earliest flutes had four or five notes from the modern major scale (the latter having the same pitches as a pentatonic scale). The most recent flutes had either a major or mixolydian (major with a lowered seventh pitch) scale configuration. One flute had a misplaced hole corrected with a smaller hole placed next to it. There is also evidence of the use of a standardized pitch system by the end of this period. A strangely configured flute with the same carbon-dated age was found near the first site. This flute had holes placed in a manner not suitable for fingers; multiple tuning configurations were placed around its perimeter. It was most likely used as a template for tuning various sized flutes.[viii]

It was known that melodic instruments existed during this time period, but these findings have changed the timeline previously theorized for the development of music. More evidence of the major scale being standardized before Ancient Greece was found in Egypt. Although not as ancient as the Neolithic bone flute, this evidence has a direct link to the person thought to have "discovered" the major (also known as Pythagorean) scale. Since Pythagoras himself spent much time in Egypt, their society's knowledge most likely aided his work.[ix]

A team of researchers in Egypt recently decided to analyze the tonal structure of four flutes from ancient Egypt on display in museums. Their goal was to find out if the Egyptian culture used a diatonic scale. These flutes were no longer playable, but they were able to recreate these bamboo instruments by measuring the artifacts and analyzing depictions of musicians in hieroglyphics left by Egyptians from the same time period.

Three out of the four flutes played notes from the diatonic scale; two of them played all seven pitches. The oldest of these three played a pentatonic scale. The fourth flute played seven distinct pitches, but these formed an Arabic scale originally thought to have originated in Persia at a later time (this scale contains quarter-tones described in the previous section). Another interesting outcome of this study was the possible standardization of pitch. The seven-note diatonic flutes' notes have frequencies within one hertz of each other, even though they were dated hundreds of years apart and located hundreds of kilometers away from one another. The pentatonic flute had a different length and therefore a different starting pitch, but seems to be based on the same standard as the other diatonic flutes.[x]

The researchers of this study admit that the results would be more compelling with the addition of more flutes for analysis. However, the evidence is strong enough to know that ancient Egypt used a diatonic scale.

CHAPTER 6

THE "PYTHAGOREAN" SCALE

As mentioned before, Pythagoras (569-475 BC) spent time in Egypt. This further complicates the traditional view that he and his followers invented the mathematical tuning methods bearing his name. Records in China indicate that a similar tuning method was developed two thousand years before Pythagoras. According to documents written about 240 BC, emperor Huang Ti (2700 BC) told a music master named Ling Lun to build a set of 60 bells. Ling Lun came up with a mathematical method for creating pitch pipes to tune this large set of bells. The "spiral of fifths" produced by Ling Lun's system is almost identical to Pythagoras's solution.[xi]

An elaborate set of bells has been discovered in China. Dated at 433 BC, these can play twelve tones and were tuned using perfect intervals. This seems to confirm a deeper understanding of music theory and chromatic tuning than even Chinese scholars had anticipated. Unfortunately, the Chinese emperor Qin Shiuangdi (ruling 221-210 BC) destroyed many music documents and instruments. Much of the evidence of China's advancements in chromatic music was probably lost at this time.[xii] Since then, Chinese music has traditionally been based on the pentatonic scale. Even after the bells were discovered in 1979, the Chinese government was slow to disclose the findings that would contradict their view of traditional music.

The mathematics described in this section accurately depicts both Pythagoras' and Lun's tuning methods (discrepancies between the two will be outlined). The system for this method of tuning is based on one interval. In music, there is a "circle of fifths." Two notes five diatonic steps apart (counting the bottom note as "one") are called a fifth. The actual number of chromatic steps is always eight. If you create another fifth from the top note of the previous fifth and continue this pattern, you will get the following series of notes:

C, G , D, A, E, B, F#, C#, G#, D#, A# (or Bb), F, C

It is called the circle of fifths because it ends where it began after the twelfth note. Using the string length ratio "2:3," a perfect fifth can be tuned very easily. By tuning either up a fifth or down a fourth (inverted fifth with the ratio 3:4), all twelve notes could theoretically be tuned in one octave. Ling Lun described this as add and subtract one-third. In other words, the next length in the sequence would be:

$x - \frac{x}{3} = \frac{2}{3}x$ with x being the length of the lower note of the interval

or

$x + \frac{x}{3} = \frac{4}{3}x$ with x being the length of the higher note of the interval

Both Pythagoras' and Lun's methods produced relatively the same results. The following chart shows the string length intervals of the chromatic scale produced by these tunings:

Comparison of Ling Lun's and Pythagoras' Tunings

	C	C#	D	Eb (D#)	E	F (E#)	F#	G	G#	A	Bb (A#)	B	C
Ling Lun	1:1	2048:2187	8:9	16384:19683	64:81	131072:177147	512:729	2:3	4096:6561	16:27	32768:59049	128:243	1:2
Pythagoras	1:1	2048:2187	8:9	27:32	64:81	3:4	512:729	2:3	4096:6561	16:27	9:16	128:243	1:2

The differences found between these two representations are only due to the starting and ending points of the “circle.” Lun’s version starts at “C” and continues to “E#” (not “F” since it was calculated as a fourth below “A#”) and the Pythagorean version starts at “C” going in two directions and ending at “Eb” and “G#.” It must be noted that this comparison is difficult to present since it involves using modern terms for note values and comparing two theories with their own representations of pitch. Hence, this comparison is very likely skewed by modern interpretations. Regardless of any possible notation discrepancies, both systems are remarkably similar.

As well as sharing many of the same ratios, both of these scales also contain the same flaw. The point where the “circle” closes does not form a perfect fifth. This is located between “E#” and “C” in the Chinese system and “Eb” and “G#” in the Greek system. The imperfect fifth was later named a “wolf” interval and became the focus of future musicians and mathematicians from the Middle Ages until today.

Evidence suggests the Chinese understood this to be a spiral of fifths and that this method would produce an infinite number of pitches. They chose to use only the first twelve tones from this spiral. Unlike the Greeks, they built instruments with all twelve notes. The Greeks created eight note scales from the twelve tones and simply avoided the “wolf” interval. These scales existed prior to the tuning method being made public; they were only enhanced by the Pythagorean’s work.

The Pythagoreans believed that important intervals were based on string length ratios with integers one through four. The ratios within one octave are: 1:1 (unison), 1:2 (octave), 2:3 (fifth), and 3:4 (fourth). Although the Pythagorean scale is built using these ratios, the simplicity in harmonic intervals is lost in the final product. As instrumental music became more complex, this scale did not have a sufficient number of "consonant" intervals (based on simple ratios). With help from rediscovered writings of a scholar from Alexandria, a less systematic but more "perfect" scale was created during the Middle Ages in Europe.

CHAPTER 7

JUST INTONATION

When musicians play an instrument with a flexible pitch system (e.g. voice or violin), it can be argued that they perform in "just" intonation. Pythagoras was correct in stating we prefer simple intervals. It does not matter to the human brain when these intervals are not the same throughout all possible key signatures. Just like a driver making slight adjustments to a steering wheel to keep a car between the lines, musicians can adjust the frequencies of a pitch when facing different contexts.

The Pythagorean focus on fourths and fifths was shared by early European musicians in terms of harmony (more than one note played at once). The more simple an interval, the less tension is heard when this interval is played at once. Starting in the middle ages, monks in the Roman Catholic Church sang Gregorian chants during worship. The melodic structure of these chants was based on ancient writings about the Greek tetrachords (sets of four notes paired to make an eight note scale). To this day, Western musicians use the Greek words for these "modes" (scale configurations). However, the names do not match the original intent of the Ancient Greeks because of interpretative mistakes.[xiii]

These Gregorian Chants are still performed today, but they also acted as a starting point for developments in polyphony (more than one melody played at once). As hundreds of years passed, church composers began experimenting with harmonic variations of the Gregorian chant. Starting with simple one-note drones behind a more complex melody (organum) and eventually moving to parallel melodies with a consonant distance apart (discant). By the thirteenth century, the increasing complexity of European sacred music was also influencing secular music[xiv].

Many fixed pitch instruments were shunned during the early Middle Ages because of their connections to "pagan" traditions of Rome. Versions of these instruments slowly came back however. Bell sets could be found throughout the middle ages. Large pipe organs were built in Cathedrals starting in the tenth century. The percussive dulcimer was not a new instrument, but was introduced to Western Europe (invented in Persia) by the fourteenth century. The psaltery (a small, hand plucked predecessor to the harpsichord) was another fixed pitch instrument introduced from the Middle East and commonly played during the later part of the Middle Ages.[xv]

The most common tuning in the first part of the Middle Ages was the Pythagorean.[xvi] The combination of increased harmonic complexity and the popularity of fixed pitch instruments made this a more difficult tuning to use. Most combinations of notes from the Pythagorean result in complex length ratios. Complex length ratios produce a discordant sound when played together. The only combinations of notes producing pleasant harmonies are all but each of the fourths and fifths. Having human voices demonstrate the ideal harmonies throughout the scale probably highlighted flaws for listeners.

Writings by the Alexandrian philosopher Claudius Ptolemy (85-165 CE) influenced the development of a new tuning system during the sixteenth century in Europe. In *Harmonics*, Ptolemy expanded the consonant string length ratios to contain whole numbers up to six. Ptolemy's tetrachords balanced the debate between those who believed scales should be based on mathematical proportions and those who thought they should be completely based on what is aesthetically pleasing.[xvii] Although less symmetric than the Pythagorean tuning, his were based on simple ratios that were also pleasing to the ear.

By the early Renaissance, musicians began to favor thirds and sixths as harmonies. The Pythagorean explanation of consonance did not include these intervals, but Ptolemy's did. A major third (five semitones) and major sixth (ten semitones) can be produce by a string length ratio of (4:5) and (3:5) respectively. Italian music theorist Gioseffe Zarlino (1517-1590) studied the intervals discussed by Ptolemy and found that the most consonant intervals were created from ratios derived from the expression: $(n+\frac{1}{n})$.[xviii] Matching these types of ratios with intervals considered to be consonant during this time, Zarlino published a standardized system for tuning an octave with "just" intervals.[xix] Although not the basis for its development, the diatonic notes (naturals) in Zarlino's Just Intonation scale are simple harmonics found in the note "F," and the accidentals (sharps and flats) are simple harmonics of various diatonic notes. The following chart displays the diatonic ratios.

Just Intonation Diatonic String Ratios

	C	D	E	F	G	A	B	C
Just Intonation Ratios for Scale	1:1	8:9	4:5	3:4	2:3	3:5	8:15	1:2
Ratios between major and minor seconds	8:9	9:10	15:16	8:9	9:10	8:9	15:16	

With this configuration, there are two types of major seconds (two semitones apart) and one type of minor second (one semitone apart) in the major scale. Further analysis would reveal more than one type of other intervals as well. Adding the chromatic ratios highlights the unevenness of this scale.

Just Intonation Chromatic Ratios

	C	**C#**	**D**	**D#**	**E**	**F**	**F#**	**G**	**G#**	**A**	**A#**	**B**	**C**
Just Intonation Chromatic Ratios	1:1	15:16	8:9	5:6	4:5	3:4	32:45	2:3	5:8	3:5	5:9	8:15	1:2
Ratios between minor seconds	15:16	128:135	45:48	24:25	15:16	128:135	15:16	15:16	24:25	25:27	72:75	15:16	

The two tuning methods discussed so far became the extremes from which mathematicians and music theorists either tried to find compromises or to perfect. The Pythagorean was developed using a systematic and uniform method. Searching for pleasing harmonies and not using any sort of algorithm was the basis for just intonation. These two philosophies were later brought closer together with the discovery of overtones.

The Greeks wrote about the harmonic series for interval string lengths, but not for multiple pitches simultaneously produced on one string. The first publication of this knowledge was by French mathematician Marin Mersenne (1588-1648 CE). Although he could never explain the physics behind them, Mersenne was the first to identify overtones found in a vibrating string.[xx] The set of string lengths needed to produce these overtones as fundamentals is the harmonic series. A mathematically predictable property of sound could be used to identify consonance of intervals. Instead of identifying consonant ratios by trial and error, one could compare the frequencies of each string length and find out how many overtones are shared. The fewer audible frequencies in common, the more dissonant the interval is for two string lengths.

While helping to identify consonance in a more scientific way, more questions were created by this discovery. For example, the most consonant interval (other than an octave) is a perfect fifth. This matches music theory from ancient Greece through the Middle Ages, but musicians began to favor major thirds by the early Renaissance. This contradiction allowed continuation of the debate between those who sought a mathematical answer and those who favored a more observational approach. Also, knowledge of the overtone series made it clear that any tuning of a fixed pitch instrument would be a compromise of ideal harmony. Before equal temperament became the standard for Western music, two families of compromises existed: “mean-tone” and “well” temperament.

CHAPTER 8

MEAN-TONE TEMPERAMENT

In previous sections of this paper, tunings have been described by string length ratios. By the Renaissance, frequency ratios were the common way to define tunings. The conversion from one to the other is simple; the reciprocal of one equals the other. For example, a perfect fifth has a string length ratio of (2:3) and a frequency ratio of (3:2). In keeping with the writings of those who developed the following tuning methods, frequency ratios will be used to describe the remaining tuning systems.

Mean-tone temperament is a modification of the Pythagorean scale. When tuning using the Pythagorean method, the fourth note tuned from "C" is "E," which should be a major third. As mentioned before, music during the Renaissance favored major thirds over perfect fifths as harmonies. The aim of mean-tone temperament was to "purify" the major third. The Pythagorean "E" had a frequency ratio of (81:64). The "just" ratio for "E" is (5:4). The difference between these ratios became known as the *syntonic comma* (also known as the *Ptolemaic comma*) with a ratio of (81:80). This comma is large enough to be audible to most listeners.

If perfect fifths are tuned flatter or sharper than (3:2), other intervals in the Pythagorean method can be made closer to "just" intervals. If fifths are tuned one-fourth of a syntonic comma flat: $\left(\frac{3}{2} \div \left(\frac{81}{80}\right)^{\frac{1}{4}} = \frac{80^{\frac{1}{4}}}{2} \approx 1.495348781\right)$, the note "E" will have a ratio of

(5:4). This creates a new scale that weakens fifths to strengthen a major third. The following is a chart of the approximate decimal values of a scale tuned this way compared to “just” intonation.

Quarter Comma Meantone Temperament vs. Just Intonation

	C	Db	D	Eb	E	F	F#	G	G#	A	A#	B	C
Meantone	1.0000	1.0700	1.1180	1.1962	1.2500	1.3375	1.3975	1.4953	1.5625	1.6719	1.7469	1.8692	2.0000
Just	1.0000	1.0417	1.1250	1.2000	1.2500	1.3333	1.4063	1.5000	1.5625	1.6667	1.8000	1.8750	2.0000
Difference	0.0000	-0.0283	0.0070	0.0037	0.0000	-0.0041	0.0087	0.0047	0.0000	-0.0052	0.0531	0.0058	0.0000

The name “mean-tone” comes from the major second (*D*) being the geometric mean of the root and the major third (*C* and *E*).[xxi] This method does not resolve the problem of having two ratios per chromatic. Identifying which combination divides the octave most evenly is the deciding factor for which sharp and flat ratios are use for chromatics. In the above chart, the note names identify which combination was chosen for this version. The combination also makes the “wolf” interval spread out between multiple pairs of notes. This first documented mean-tone tuning was known as “Quarter Comma Mean-tone Temperament” and was first published by Italian music theorist Pietro Aaron (1490 – 1545 CE) in the early fifteen hundreds.[xxii]

Musicians using this scale would not stray too far from the key of “C” (i.e. keys with too many flats or sharps) because of the more arbitrary way accidentals were tuned. Mathematical theory is not the sole basis of mean-tone temperament. Compromising the ratio of a perfect fifth in order to make a “just” third is a decision guided by aesthetics. This decision overcorrected the fifth causing problems with harmonies other than major thirds. Many other mean-tone temperaments were developed with different divisions of the syntonic comma. These will be included for analysis in this research.

CHAPTER 9

WELL TEMPERAMENT

The most common well temperament tuning was published by Andreas Werckmeister (1645 – 1706 CE) and referred to as “Werckmeister III”. As the name implies, Werckmeister created multiple versions of the well-tempered scale. His first two tunings were actually versions of Just Intonation.[xxiii] At the same time these scales were being published, composers were experimenting with modulations between keys signatures. Contrary to popular belief, J.S. Bach’s Well Tempered Clavier was not written because the scale allowed for even sounding key signatures (equal temperament is needed for this); Bach’s work actually highlighted unique characteristics for each key signature when fixed pitch instruments were tuned using well temperament. However, it is unknown which version Bach used.

Like mean-tone temperament, portions of a comma are used to alter fifths in Well-Tempered scales. However, the Pythagorean comma replaces the Syntonic. This comma is calculated by dividing the ratio of twelve consecutive perfect fifths by the ratio of seven consecutive octaves $\left(\frac{531441}{4096} \div \frac{128}{1} = \frac{531441}{524288} \approx 1.013643265\right)$. The ratio not being (1:1) is the reason for the “Wolf” fifth in the Pythagorean tuning. Another difference with Werckmeister’s tuning is the seemingly arbitrary nature in which the comma is distributed between the fifths. C:G, G:D, D:A, and B:F# are all one-fourth of a comma flat with the remaining fifths tuned perfectly.[xxiv] The chart below shows the frequency ratios in decimal form.

Werckmeister III Well-Temperament

	C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C
Frequency ratios	1.0000	1.0535	1.1174	1.1852	1.2528	1.3333	1.4047	1.4949	1.5802	1.6704	1.7778	1.8792	2.0000
Ratios between chromatics	1.0535	1.0607	1.0607	1.0571	1.0643	1.0535	1.0643	1.0571	1.0571	1.0643	1.0571	1.0643	

An advantage to this tuning is having one value for each chromatic. The spiral of fifths becomes a circle since four of the fifths tuned are lowered by one-fourth of a comma. All of the well-tempered scales share this quality. The disadvantage is the unevenness of semitones. The consonance of commonly used intervals is compromised for certain keys more than others.

The way in which the Pythagorean comma is spread throughout the circle of fifths is what makes each well temperament unique. Analyzing their harmonic strengths would most likely correlate with the commonly used harmonies and key signatures of the time and place of their invention. With composers expanding the possibilities for key modulations and harmony, it became increasingly difficult to create a tempered scale for fixed pitch instruments. The scale used today was the compromise reached to accommodate the developments in Western music.

CHAPTER 10

EQUAL TEMPERAMENT

Many mathematicians and music theorists considered the equal temperament scale well before the first accurate table of string lengths for it was published. Those working on other temperaments saw the advantage of spreading imperfections over multiple fifths rather than having one interval completely out of tune, so the next logical step was extending this to all fifths. Galileo's father, Vincenzo Galilei (~1525-1591 CE), wrote about the placement of frets on a lute in 1581. Fretted instruments have more complications in temperament because the parallel strings (usually tuned a perfect fourth apart) have identical string length ratios throughout their ranges due to the perpendicular frets on the fingerboard. Galilei proposed that the only temperament that will work with this setup is one that divides the octave equally. He gave an approximation of this division with each semitone frequency ratio of (18:17).[xxv] This ratio does not divide the octave equally, but it proved to be very close to the solution described below.

Until the seventeenth century, no practical and accurate application of equal temperament was published in Europe. This was not due to lack of inspiration on the part of those trying to develop a tuning method; the barrier was directly related to an unsolved mathematics problem. The classical problem of doubling a cube is one that Greek mathematicians had tried to solve for centuries. In the nineteenth century, French mathematician Pierre Wantzel (1814-1848 CE) proved that the geometric solution they were attempting was impossible using only a ruler and compass.

The cube root of two is more than just irrational; it cannot be constructed with a ruler and compass.[xxvi] Spreading the error over twelve fifths of a Pythagorean tuning involved splitting the error into twelve equal parts. Since these are proportional relationships, this meant taking the twelfth root. Three is a factor of twelve, and the cube root of a Pythagorean comma is part of the calculation of values of an equal tempered scale. Since the denominator of $\left(\frac{531441}{524288}\right)$ is a power of two, the cube root of two is crucial in calculating string ratios.

A simpler way of illustrating this problem is to ignore the Pythagorean comma and create a scale based on the desired result. All semitones should have the same proportion because all intervals are consistent in an equal tempered scale. Twelve of these semitones will equal one octave with the frequency ratio of (2:1). Therefore, a semitone must have a ratio of $(2^{\frac{1}{12}}:1)$. Already, the cube root of two is in the answer for this tuning. The following shows all of the ratios and decimal values for an equal tempered scale.

Equal Temperament

	C	**C#**	**D**	**D#**	**E**	**F**	**F#**	**G**	**G#**	**A**	**A#**	**B**	**C**
Frequency Ratios (all to one)	1	$2^{\frac{1}{12}}$	$2^{\frac{1}{6}}$	$2^{\frac{1}{4}}$	$2^{\frac{1}{3}}$	$2^{\frac{5}{12}}$	$2^{\frac{1}{2}}$	$2^{\frac{7}{12}}$	$2^{\frac{2}{3}}$	$2^{\frac{3}{4}}$	$2^{\frac{5}{6}}$	$2^{\frac{11}{12}}$	2
Approximate Decimal Values	1.0000	1.0595	1.1225	1.1892	1.2599	1.3348	1.4142	1.4983	1.5874	1.6818	1.7818	1.8877	2.0000

The advantage to this is clear; all keys signatures sound exactly the same. If people can accept the sound of one major key in equal temperament, all major keys will sound satisfactory to their ears. The same could be said for other modes, such as those used in Gregorian chants. Unfortunately, this could not be tested until the values for each chromatic were known. Since the tuning contains the cube root of two, the scale had to wait until mathematicians were capable of accurately estimating this value.

Dutch mathematician Simon Stevin (1548-1620) attempted to calculate the equal tempered scale during the 1580's. In his writing, he did not refer to it as temperament. He believed equal proportions of the octave produced the ideal scale, and all other tunings were misconceptions. To make his calculations, he first established a value for "E" by estimating the cubed root of two. He used this value and the properties of proportions to calculate values for the remaining notes. Although he had many errors due to improper rounding of decimals, his table of values produces a tuning that resembles the equal temperament used today.[xxvii]

Because Stevin's work was never published, equal temperament was not experimented with until after 1636 when Marin Mersenne published various calculations of equal temperament in his *Harmonie Universelle*. The most accurate of these calculations was given to Mersenne by French engineer Jean Gallé (1580-? CE). With the exception of one (probably typographical) error, these calculations are indistinguishable from the current version. Mersenne is often referred to as the inventor of equal temperament with Stevin or Gallé hardly mentioned. Although Stevin was the first Western European known to tune using equal temperament, it is possible that Mersenne and Stevin found this method from China.

Chinese prince Chu Tsai-Yu published an equal temperament scale in 1584. His calculations were as accurate as those done by Gallé fifty years later. It would seem unlikely to have influenced Europe so quickly; however, Italian Jesuit Matteo Ricci (1552-1610 CE) went to China to start a mission in 1583. Although religion was part of his mission, his main focus was sharing Western mathematical and scientific methods with Chinese scholars. He became a well-respected figure in their country. Trying to gain the respect of Chinese royalty, Ricci was most likely aware of Chu Tsai-Yu's publication. There is no evidence of Ricci sending the publication to Europe. Whether or not he directly influenced Western European music, Chu Tsai-Yu is the first person known to have made an accurate table for tuning in equal temperament.[xxviii]

Another mathematical advancement contributing to equal temperament was the logarithm. Scottish mathematician John Napier (1550-1617 CE) created a calculation table. His table was later updated by Henry Briggs (1561-1631 CE) to reflect changes planned by Napier shortly before his death. Briggs' update (published in 1628 CE) became the form of logarithms used today.[xxix] Thanks to this new tool, calculations for frequencies could be made more accurately and quickly without the time-consuming extraction of roots. Taking logarithms of frequency ratios (base two) and multiplying them by the number (1200) later became the standard for measuring pitch called *cents*. This measure is beneficial because it is small enough to measure important intervals such as the Pythagorean comma (approximately 23.46 cents), and one hundred cents represent a semitone in equal temperament. Alexander Ellis (1814-1890 CE) first popularized this measure in his English translation of Hermann Helmholtz's (1821-1894 CE) *On the Senations of Tone.*[xxx] The following chart shows each of the values for an equal tempered scale, followed by the logarithm and cents for each of these values.

Equal Temperament with Logarithms (base 2) and Cents (log *1200)

	C	**C#**	**D**	**D#**	**E**	**F**	**F#**	**G**	**G#**	**A**	**A#**	**B**	**C**
Frequency Ratio	1	1.0595	1.1225	1.1892	1.2599	1.3348	1.4142	1.4983	1.5874	1.6818	1.7818	1.8877	2.0000
Log of Ratios	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{11}{12}$	1
Cents	0	100	200	300	400	500	600	700	800	900	1000	1100	1200

Although this new standard of measure was obviously inspired by equal temperament, logarithms advanced the study of all temperament methods. As can be seen by the dates for methods mentioned so far, there were overlaps due to conflicting opinions about which method was better. Even Mersenne invented his own modification of mean tone temperament after publishing the equal tempered scale.[xxxi] Equal temperament did not become the dominant tuning method until the end of the eighteenth century.[xxxii] Logarithms (and later cents) became a convenient way to calculate and express frequency values for the many versions of mean-tone and well temperament. The analysis in the next section will use cents to express the frequencies of all tuning methods including the Pythagorean.

By the nineteenth century, Western European music had become so complex that equal temperament was the only practical method for tuning fixed pitch instruments. Contrary to Stevin's view of the scale, most saw this as a necessary compromise. Other temperaments favored particular key signatures. Musical compositions were created in all twenty-four major and minor keys by this time. As well as changing keys mid-piece, harmonic progressions often mixed chords from other key signatures with the diatonic chords. Without equal temperament, fixed pitch instruments would have needed retuning constantly (even in the middle of a performance) to facilitate these new compositional techniques. In the next section, selected common tunings leading up to this acceptance of equal temperament will be statistically analyzed.

CHAPTER 11

DATA ANALYZED IN THIS STUDY

The main question this analysis seeks to answer is: Did the acceptance of the equal tempered scale fit into an evolutionary development or was it a radical departure from previously accepted methods? The country of origin for each tuning is included in the analysis because of the influence this variable might have. However, the study does not seek to answer any questions regarding the origin of each tuning method; the variable is only used to identify or account for any disparity in the data resulting from the variety of locations.

The tunings used for this research come from various sources, with the majority from *Tuning and Temperament* by J. Murray Barbour.[xxxiii] Barbour used cents to analyze many historic temperaments. He also included references to the initial publications of these methods. Cross-referencing this data with tables found on the Internet, over ninety tunings with dates and countries of first publication are compiled for this research (Appendix A). Equal temperament attempts were not included for this study.

There is no method to identify what tuning methods were prevalent for any given time or place. Debate is still ongoing concerning which scale Bach's *Well Tempered Clavier* was designed for. If records are insufficient for a piece whose title alludes to a family of tuning systems, one can imagine the difficulty in accurately pinpointing the original temperament for other pieces. In this study, tunings are ordered and placed by their dates and locations of first publication. This is not an accurate portrayal of the music heard during those times, but theorists and mathematicians were probably influenced by the current practice and vice versa. Rather than skewing the results by trying to assess which tunings were important (based on modern notions), no documented twelve note tuning system from Europe was purposely excluded from this data set.

The pitch systems were first analyzed by mean and standard deviation from three chosen standards: Equal Temperament, Just Intonation, and Pythagorean. These were chosen because they are all somewhat static and are historically significant. Total deviation values for a scale were calculated by adding the absolute value of differences between cent values of each note and the corresponding cent values from the standard. This sum was then divided by eleven to obtain the mean deviation (see example below). Eleven was used instead of thirteen because the first and last notes never change.

Example Calculation of Mean and Standard Deviation (using Silbermann w/ET as standard)

	C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C
Analyzed Scale (Silberman)	0	89	197	305	394	502	590	698	787	895	1003	1092	1200
Standard (equal temperament)	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
Absolute value of difference	0	11	3	5	6	2	10	2	13	5	3	8	0
$(x-\bar{x})^2$ **with corresponding cent values as** $\bar{x}$ **.**	0	121	9	25	36	4	100	4	169	25	9	64	0

Sum of Deviations	68	**Mean of Deviations**	6.182	**Standard Deviation**	$\sqrt{\frac{\sum(x-\bar{x})^2}{(n-1)}} \approx 7.523$

Standard deviations were calculated with the usual formula, replacing means with corresponding cents from the standard (see above). Eleven was once again used for the number of samples. Mean and standard deviations were listed and referred to in Barbour's text, but he did not explain the calculations used to arrive at them. Therefore, the deviation figures used for this research are not the same as those listed in Tuning and Temperament.

CHAPTER 12

METHODS AND RESULTS OF THE ANALYSIS

The mean and standard deviation values were the primary focus of this analysis. If either of these values had decreased over time, this would indicate a trend towards the given standard. The opposite would be true if these values had increased over time. Since the question to be answered relates to the existence of an evolution towards equal temperament, this tuning was the most important standard used. The other two standards were included to see if a different type of evolution preceded the acceptance of equal temperament.

Since all of the data sets contained multiple outliers, different transformations were experimented with. Square root transformations worked the best for most, but the just intonation standard still contained multiple outliers. Linear regressions with the dates of publication as a predictor did not produce convincing results. The best of all these regressions is the mean deviation with equal temperament as the standard (p-value of .001 w/DF 1). This regression has an R-square value of (8.5%), which is not enough to prove any correlation. The R-square value improved to (33%) after adding significant binary categorical variables to represent the countries in which each tuning was printed (found by stepwise regression). The null hypothesis for the "date" coefficient was not refuted for this regression (p-value of .106 w/DF of 3). In other words, adding the location of publications to the prediction made the date of publications statistically insignificant. Regardless, the R-square value was still too low to prove any correlations. (see Appendix B for all tests and graphs mentioned in this paragraph)

A possible problem with the analysis could be the repetition of dates. Some theorists and mathematicians published multiple methods in one publication, particularly in the seventeen hundreds. They often included theories that would span a wide range of deviations. This becomes evident when viewing a regression plot of the equal temperament mean deviation versus time (Appendix B). To eliminate this factor as a possible "smokescreen" hiding trends, the data table was reduced by selecting only the first two theories from any publication. There was no effort to select only "important" tunings.

With this data set, the residuals are more evenly spread out versus fits. The R-square value of the "date" regression for this modified set is (2.2%). The R-square value for stepwise regression with locations added is also lower than the first (17.6%). The null hypothesis is not refuted for any of the coefficients except the binary variable for Germany (p-value of .001 w/DF 2). No trends in the data could be identified with any regression models described so far. Curve fittings were also experimented with, but with similar results (Appendix C).

Given the lack of any trends found by regression, classification methods were next experimented with. Many divisions of the population were tried. Dividing the time period into four groups (each about 92 years long) was the selected division as it produced the most significant results. These quarters contain unequal numbers of cases, but the subgroup variances for equal temperament deviations (both mean and standard) are acceptable for an ANOVA. The subgroup variances are too varied in size for the other standards (Just and Pythagorean), and non-parametric tests such as Mood's median must be run when including these variables (Appendix D).

ANOVA's and Mood's median tests were preformed for each combination of standards and deviation by quarter. The only two tests to show a distinction between quarters were both equal temperament standards. Although the null hypothesis is refuted for these tests (p-value of .001 w/ DF of 92 for both tests), not all quarters are distinct from each other when considering the (95%) confidence intervals (the standard deviation test is shown below).

One-way ANOVA: Standard Deviation from ET versus Quarters

```
Analysis of Variance for sqrtET_S
Source     DF        SS        MS        F        P
Quarters    3    16.005     5.335     5.68    0.001
Error      89    83.662     0.940
Total      92    99.668
                                   Individual 95% CIs For Mean
                                   Based on Pooled StDev
Level       N      Mean     StDev  ------+---------+---------+---------+
1           8    3.4325    0.7710              (----------*-----------)
2          15    3.5913    0.9847                    (-------*-------)
3          27    2.4148    0.9394  (-----*-----)
4          43    2.8249    1.0116          (----*----)
                                   ------+---------+---------+---------+
Pooled StDev =   0.9696                2.40      3.00      3.60      4.20
```

The largest difference is between the second (1533-1623) and the third (1624-1715). The third quarter has a dramatic shift closer to equal temperament with the fourth quarter returning closer to the second. Unfortunately, the data could be skewed because of the previously mentioned problem of multiple tunings per publication. Running the same test using only the first two tunings of each publication results in the following:

One-way ANOVA: Standard Deviation from ET versus Quarters with Two Tunings per Publication

```
Analysis of Variance for sqrtET_S
Source     DF        SS        MS        F        P
Quarters    3     2.209     0.736     1.05    0.379
Error      60    42.249     0.704
Total      63    44.457
                                   Individual 95% CIs For Mean
                                   Based on Pooled StDev
Level       N      Mean     StDev  --+---------+---------+---------+----
1           8    3.4325    0.7710     (----------------*----------------)
2          15    3.5913    0.9847              (------------*-----------)
3          10    3.2660    0.8580  (--------------*--------------)
4          31    3.1410    0.7722     (--------*-------)
                                   --+---------+---------+---------+----
Pooled StDev =   0.8391            2.80      3.15      3.50      3.85
```

With the p-value shifting from (.001) to (.379), it is obvious that the repeated values for dates have a significant impact on any analysis. Removing cases from the repeated dates is not an appropriate answer to this problem. It assumes an importance of the first two tunings of each publication and excludes numerous repeated values from important theorists such as Werkheimer and Neidhardt. Therefore, tests were run on a new set of data containing the average of deviations from each publication (Appendix E). This insures that no dated publication is given more weight than others and all tunings from each publication are represented. Some theorists are counted two or three times because of multiple publications. However, these publications have separate dates and can actually help in identifying trends.

The new data is more evenly spread than the first, but a square root transformation still improved the normality and brought the variances closer together in size. The Just Intonation numbers contain multiple outliers for mean deviation regardless of transformations and the Pythagorean has one outlier for each of its graphs. The dates are distributed in a more normal pattern, and are linear when presented in a series plot (Appendix F).

Once again, linear regressions were run on each of the standards. Using the date as a predictor, none of the standards produced a regression with coefficients disproving the null hypothesis (Appendix G). The only regression that contained significant coefficients was one predicting the date using a combination of the mean deviations of Pythagorean and Equal temperament standards. The R-square value for this regression is (16.9%). Adding the country data raises this R-square value. The most successful combination is below (full output can be seen in Appendix G).

```
The regression equation is
Date = 1696 - 98.2 ET_MD_SR + 90.1 PTH_MD_SR - 140 Italy - 212 Spain
           - 84.0 Germany

Predictor        Coef     SE Coef          T        P
Constant      1696.15       58.83      28.83    0.000
ET_MD_SR       -98.17       22.35      -4.39    0.000
PTH_MD_S        90.11       17.19       5.24    0.000
Italy         -140.46       33.25      -4.22    0.000
Spain         -212.47       47.65      -4.46    0.000
Germany        -84.03       27.81      -3.02    0.004

S = 75.17       R-Sq = 56.6%     R-Sq(adj) = 50.9%
```

Despite these stronger values, there are problems with the model. Removing three samples with extreme residuals alters the regression to one with insignificant tuning variables.

```
Date = 1785 - 112 ET_MD_SR + 79 ET_SD_SR - 103 Italy - 176 Spain -
68.3 Germany

Predictor        Coef     SE Coef          T        P
Constant      1785.16       74.87      23.84    0.000
ET_MD_SR       -112.0       228.6      -0.49    0.627
ET_SD_SR         79.1       200.9       0.39    0.696
Italy         -102.76       42.53      -2.42    0.021
Spain         -175.73       61.75      -2.85    0.007
Germany        -68.30       37.08      -1.84    0.073

S = 98.48       R-Sq = 25.6%     R-Sq(adj) = 15.8%
```

Given the instability of this regression, classification methods were tried with the averaged data set. Both transformed equal temperament variables grouped by quarters passed the Levene and Bartlett tests for equal variance. The transformed mean deviation of just temperament barely passed these tests with p-values just above (.05) and nonparametric tests will be used for this variable. The transformed standard deviation of just intonation deviation was more acceptable. Both Pythagorean data sets were closer in variance with the untransformed data (Appendix H).

Based on the characteristics listed above, the appropriate tests (ANOVA or Mood's median) were applied to the appropriate data (transformed or original) for each variable. Every test failed to refute the null hypothesis (Appendix H). Therefore, there is no distinction between any of the deviations when grouped into four quarters of the time period in question.

Multivariate tests were done with both data sets mentioned in this section. These were generally used to explore the data before running the tests already described. The first two tests were stepwise discriminant analyses run on the data as it was originally entered. All transformed deviation standards were included for these tests. The second test also included the binary country of publication data. The first test produced two discriminant functions with the mean deviations of the equal tempered and Pythagorean data. Only (46%) of cases were correctly predicted with this function. The second stepwise discriminant analysis automatically removed all variables regarding pitch (Appendix I). Similar discriminant analyses were attempted on the averaged data, but all variables (unaltered and transformed) were eliminated from any test (Appendix J). Although redundant, these multivariate tests confirm the results of the other tests described in this section.

CHAPTER 13

INTERPRETATION OF THE RESULTS

It is harder to prove that something does not exist than to prove that it does. This analysis did not find any proof of an evolution leading to equal temperament. If one existed, it would likely be reflected in the writing of theorists and mathematicians. Even the tests that came close to identifying a trend did so in a manner that is counterintuitive to what was being searched for. For example, the regression based on averaged data predicted that equal temperament would occur earliest in Spain in the year 2168 CE. Even if this were a valid test, equal temperament becoming the standard would have been a radical change in the nineteenth century.

As well as not finding any trends, dividing the tunings into groups of time failed to find a statistically significant difference between any two quarters. The first and last quarters were not distinct even though they were separated by almost two hundred years. This is not to say tuning methods did not change. A dot plot of the deviations from equal temperament over time shows an oscillating pattern, occasionally coming close to equal temperament (Appendix K).

Many of the theorists and mathematicians mentioned in this study were familiar with the concept of equal temperament. Many of them were also capable of the root extraction or logarithm methods for accurately finding equal tempered measurements. Those who favored well temperaments sometimes came close to our current tuning.

Time series plots featuring individual well temperament advocates (who published more than two tunings) show that they often approached equal temperament and then strayed away from it in later tunings (Appendix K). For example, Neidhardt's *Third Circle Number Five* came very close to equal temperament (mean deviation of .7 cents and standard deviation of 1.3 cents). Twenty-six years later, Neidhardt published his *Sample Number Two* tuning, which was a drastic shift away from equal temperament (mean deviation of 8.0 cents with a standard deviation of 9.1 cents). This could be due to a conflict between mathematical perfection and aesthetics.

It is possible that authors presenting multiple scales per publication used most of them as examples to refute, while promoting one scale. The references used for this research presented each of these scales as equally important, but reading and analyzing each of the original publications is the only verification that all scales included for analysis represent the true intents of each author. Therefore, future research is needed to substantiate the findings of this thesis. The standards and analysis outlined in chapters eleven and twelve could be reapplied to a filtered version of the data set if any of the included scales were in fact "straw men." Given the obscure nature of the publications in question, this could prove to be a long and arduous process.

CHAPTER 14

CONCLUSION

Specifically answering the question for this thesis, equal temperament was a radical change in tuning. However, the analysis also showed that every tuning could be described as a radical change from its predecessor because no trends were evident. Equal temperament could be seen as an acquired taste with its adoption as a standard much later than its invention. It could also be seen as a last resort, saved until music surpassed the limitations of all other possible methods. Either way, favoring a tonality system in the eighteenth century and keeping it as a standard for over two hundred years is a radical change considering the amount of variability throughout the previous three centuries.

With the knowledge that our current tuning system is mathematically based, it is interesting that mathematicians often invented other systems. For example, mathematician Leonard Euler (1707-1783 CE) invented a tuning method in 1739 CE (103 years after Mersenne's *Harmonie Universelle*). This method was a version of "just" intonation and deviated far from equal temperament (mean deviation of 15.3 cents and standard deviation of 19.1 cents). Since Euler wrote extensively on the mathematical evaluation of consonance and "just" intonation is built from consonant intervals, it is not surprising he experimented with this tuning.[xxxiv] He later became an advocate for equal temperament, stating that his theories of consonance still held true because the difference between the tempered ratios and the ratios studied in his work were "almost imperceptible." [xxxv]

This highlights the conundrum for those who tried to perfect the tuning system for fixed pitch instruments. Different types of mathematical perfection are all possible, just not at the same time. Some mathematicians and musicians focused on the perfection of the interval and harmony itself, while others focused on expanding the possibilities for key signatures and modulation. The fact that these two searches lead to very different answers explains the variability in tunings throughout the time period analyzed by this research.

This study only provides evidence that the transition to equal temperament was not a natural one, but was driven by necessity. It does not answer any questions as to the impact of equal temperament on our society. Many of those currently arguing against the use of equal temperament look to the past for the "ideal" tuning method, but just as early composers wrote music within the context of historic tunings, today's composers write for equal temperament. For this reason, it is hard to imagine going back to an earlier method.

Equal temperament was a compromise that took almost two hundred years for Western civilization to accept for fixed pitch instruments. Now that generations of people have been exposed to this tonality structure, the current perceptions of consonance and dissonance might have been altered. Future studies are needed to assess the impact of equal temperament on the human interpretation of intervals and harmony. These studies can help to decide if and how the tuning of fixed pitch instruments could be improved with new technology.

With the combination of MIDI (Musical Instrument Digital Interface) and other technological developments such as self-tuning grand pianos[xxxvi], fixed pitch instruments could someday evolve into responsive dynamically tuned instruments. If this does happen, music theorists and mathematicians will probably debate the weaknesses and strengths of various programs for these new instruments. However, having been conditioned to equal temperament for two centuries, the first debate may be whether or not change is desired.

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APPENDIX A – ORIGINAL DATA SET FOR THIS STUDY

	C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C2	Date	1/4	Type	Country	ET_MD	ET_SD	JST_MD	JST_SD	PTH_MD	PTH_SD	Reference
van Zwolle (transposed Pythagorean)	0	90	204	294	408	498	588	702	816	906	996	1110	1200	1440	1	P	France	7.11	8.61	15.91	21.69	4.44	10.62	http://www.terryblackburn.us/music/temperament/#project http://www.xs4all.nl/~huygensf/doc/bib.html
Erlangen (anonymous, transposed Pythagorean	0	92	204	294	386	498	590	702	794	906	996	1088	1200	1475	1	P	Germany	6.73	8.07	10.00	15.56	10.00	15.56	Tuning and Temperament[i]
Ramis (transposed Pythagorean)	0	92	182	294	386	498	590	702	792	884	996	1088	1200	1482	1	P	Spain	9.09	10.99	9.82	15.28	14.18	18.65	Tuning and Temperament[i]
Pythagorean (published by Hugo de Reutlingen)	0	114	204	294	408	498	612	702	816	906	996	1110	1200	1488	1	P	France	7.64	9.34	20.00	26.03	0.00	0.00	Tuning and Temperament[i]
Grammateus'	0	102	204	306	408	498	600	702	804	906	1008	1110	1200	1518	1	MMT	Germany	4.73	5.87	14.55	19.49	5.45	8.49	Tuning and Temperament[i]
Meantone (-1/4 Aron)	0	76	193	310	386	503	579	697	773	890	1007	1083	1200	1523	1	MT	Italy	13.00	15.96	6.09	7.40	20.64	25.30	Tuning and Temperament[i]
Fogliano #1	0	70	182	316	386	498	568	702	772	884	996	1088	1200	1529	1	J	Italy	15.82	19.77	6.00	12.05	22.00	28.68	Tuning and Temperament[i]
Fogliano #2	0	70	204	316	386	498	568	702	772	884	1018	1088	1200	1529	1	J	Italy	15.82	19.77	2.00	6.96	22.00	28.68	Tuning and Temperament[i]
Agricola	0	92	204	296	408	498	590	702	794	906	996	1110	1200	1539	2	J	Germany	5.82	6.75	13.82	18.18	6.18	12.07	Tuning and Temperament[i]
Ganassi (just w/mean semitones)	0	88	182	281	386	498	597	702	790	884	983	1088	1200	1543	2	MMT	Italy	11.36	13.53	12.27	19.06	16.45	19.62	Tuning and Temperament[i]
Agricola	0	110	204	314	408	498	608	702	812	906	1016	1110	1200	1545	2	WT	Germany	8.36	9.92	15.27	22.32	4.73	9.21	Tuning and Temperament[i]
Bermudo's Vihuela	0	103	200	294	401	498	601	698	792	899	996	1099	1200	1555	2	WT	Spain	2.59	3.62	14.10	17.70	7.25	10.38	Tuning and Temperament[i]
Zarlino (-2/7)	0	70	191	313	383	504	574	696	817	887	1008	1078	1200	1558	2	MT	Italy	14.82	17.76	10.45	16.58	19.55	24.77	Tuning and Temperament[i]
Salinas (-1/3)	0	64	190	316	379	505	569	695	758	884	1010	1074	1200	1577	2	MT	Spain	19.82	24.29	8.91	11.30	27.45	33.63	Tuning and Temperament[i]
Schneegass (geometric)	0	79	194	309	389	504	585	698	812	892	1005	1085	1200	1590	2	MT	Germany	9.82	11.76	9.82	14.69	15.27	19.00	Tuning and Temperament[i]
Artusi #1	0	97	193	290	386	503	600	697	794	890	987	1083	1200	1603	2	MMT	Italy	7.82	9.83	13.45	17.84	13.64	16.29	Tuning and Temperament[i]

Artusi #2	0	97	197	294	394	502	599	698	795	895	992	1092	1200	1603	2	MMT	Italy	4.45	5.26	13.18	16.68	10.27	12.79	Tuning and Temperament[i]
Reinhard (just w/mean semitones)	0	99	204	292	386	498	597	702	790	884	983	1088	1200	1604	2	MMT	Germany	8.09	10.41	10.27	17.36	12.45	16.59	Tuning and Temperament[i]
De Caus	0	70	182	274	386	498	568	702	772	884	996	1088	1200	1615	2	J	Germany	16.73	20.80	9.82	17.93	21.82	28.54	Tuning and Temperament[i]
Colonna #1	0	70	182	287	386	498	568	702	732	884	989	1088	1200	1618	2	MJ	Italy	19.82	27.87	12.91	20.62	24.91	36.00	Tuning and Temperament[i]
Colonna #2	0	70	204	316	386	498	618	702	814	884	1018	1048	1200	1618	2	MJ	Italy	16.91	22.76	10.00	20.37	18.36	27.85	Tuning and Temperament[i]
Kepler #1	0	92	204	316	386	498	590	702	794	906	1018	1088	1200	1619	2	J	Germany	8.91	10.86	6.00	12.05	14.00	18.41	Tuning and Temperament[i]
Kepler #2	0	92	204	316	386	498	590	702	814	906	1018	1088	1200	1619	2	J	Germany	9.64	11.58	7.82	16.53	12.18	17.05	Tuning and Temperament[i]
Mersenne Spinet #1	0	112	182	316	386	498	610	702	814	884	996	1088	1200	1636	3	J	France	10.91	12.81	13.45	22.13	10.55	15.59	Tuning and Temperament[i]
Mersenne Spinet #2	0	70	204	274	386	498	568	702	772	884	996	1088	1200	1636	3	J	France	15.45	20.05	7.82	16.53	19.82	27.68	Tuning and Temperament[i]
Mersenne's Improved Meantone #1	0	76	193	299	386	503	579	697	773	890	1001	1083	1200	1648	3	MMT	France	11.64	15.49	7.64	9.84	19.09	24.64	Tuning and Temperament[i]
Mersenne's Improved Meantone #2	0	76	193	288	386	503	579	697	773	890	996	1083	1200	1648	3	MMT	France	12.91	15.99	9.09	12.88	18.73	24.61	Tuning and Temperament[i]
Rossi (-1/5)	0	83	195	307	390	502	586	698	781	893	1005	1088	1200	1666	3	MT	Italy	9.09	11.16	7.09	8.52	16.73	20.50	Tuning and Temperament[i]
Rossi (-2/9)	0	79	194	308	389	503	582	697	777	892	1006	1085	1200	1666	3	MT	Italy	11.09	13.63	6.91	7.81	18.73	22.97	Tuning and Temperament[i]
Werckmeister #1	0	90	192	294	390	498	588	696	792	888	996	1092	1200	1691	3	WT	Germany	7.64	8.67	10.55	14.14	13.09	16.97	Tuning and Temperament[i]
Werckmeister #2	0	90	196	294	392	498	596	694	792	890	996	1094	1200	1691	3	WT	Germany	6.18	6.99	11.27	14.28	11.64	15.18	Tuning and Temperament[i]
Werckmeister #3	0	96	204	300	396	504	600	702	792	900	1002	1098	1200	1691	3	WT	Germany	2.73	3.74	11.82	14.76	9.27	12.15	Tuning and Temperament[i]
Werckmeister #4	0	82	196	294	392	498	588	694	784	890	1004	1086	1200	1691	3	WT	Germany	8.90	10.69	8.40	10.91	15.11	19.54	http://www.terryblackburn.us/music/temperament/#project
Werckmeister #5	0	96	204	300	396	504	600	702	792	900	1002	1098	1200	1691	3	WT	Germany	2.68	3.67	11.85	14.80	9.25	12.10	http://www.terryblackburn.us/music/temperament/#project
Werckmeister #6 (1/7 comma)	0	91	186	298	395	498	595	698	793	893	1000	1097	1200	1691	3	WT	Germany	5.09	6.68	12.00	14.62	12.00	14.92	Tuning and Temperament[i]
Neidhardt's Fifth Circle #01	0	94	196	296	392	498	592	698	796	894	996	1092	1200	1706	3	WT	Germany	5.09	5.80	11.27	15.07	10.91	14.17	Tuning and Temperament[i]
Neidhardt's Fifth Circle #02	0	102	204	298	400	502	604	698	800	902	1004	1098	1200	1706	3	WT	Germany	2.18	2.68	14.18	17.82	7.27	8.94	Tuning and Temperament[i]

Neidhardt's Fifth Circle #03	0	102	200	302	400	502	600	702	800	902	1000	1102	1200	1706	3	WT	Germany	1.09	1.55	14.18	17.82	7.27	8.94	Tuning and Temperament[i]
Neidhardt's Fifth Circle #04	0	96	198	300	396	498	600	696	798	900	996	1098	1200	1706	3	WT	Germany	2.18	2.83	13.45	16.49	8.73	11.06	Tuning and Temperament[i]
Neidhardt's Fifth Circle #05	0	100	200	298	402	502	600	700	800	898	1002	1102	1200	1706	3	WT	Germany	1.09	1.55	14.18	17.46	7.64	9.21	Tuning and Temperament[i]
Neidhardt's Fifth Circle #06	0	100	196	300	400	496	600	700	796	900	1000	1096	1200	1706	3	WT	Germany	1.45	2.53	13.45	16.57	8.73	10.77	Tuning and Temperament[i]
Neidhardt's Fifth Circle #07	0	94	194	298	400	494	596	696	800	892	996	1098	1200	1706	3	WT	Germany	3.82	4.77	13.64	16.48	10.00	12.18	Tuning and Temperament[i]
Neidhardt's Fifth Circle #09	0	98	196	300	400	498	596	700	800	898	996	1100	1200	1706	3	WT	Germany	1.64	2.45	13.64	17.17	8.18	10.49	Tuning and Temperament[i]
Neidhardt's Fifth Circle #10	0	94	198	298	392	498	596	696	796	894	996	1094	1200	1706	3	WT	Germany	4.36	4.98	11.64	14.97	10.55	13.42	Tuning and Temperament[i]
Neidhardt's Fifth Circle #11	0	96	198	296	394	500	598	700	800	894	996	1098	1200	1706	3	WT	Germany	2.73	3.63	12.91	16.48	8.91	11.44	Tuning and Temperament[i]
Neidhardt's Fifth Circle #12	0	100	198	300	396	498	600	700	798	900	996	1098	1200	1706	3	WT	Germany	1.45	2.19	13.45	17.06	8.00	10.32	Tuning and Temperament[i]
Neidhardt's Third Circle #01	0	94	198	296	390	498	592	700	794	894	998	1092	1200	1706	3	WT	Germany	4.91	6.03	10.36	14.27	11.09	14.57	Tuning and Temperament[i]
Neidhardt's Third Circle #03	0	96	196	296	394	500	598	698	796	896	1002	1092	1200	1706	3	WT	Germany	3.64	4.38	12.00	15.10	10.55	12.90	Tuning and Temperament[i]
Neidhardt's Third Circle #04	0	96	196	296	396	498	596	698	796	894	1000	1094	1200	1706	3	WT	Germany	3.64	4.20	12.00	15.26	10.18	12.74	Tuning and Temperament[i]
Neidhardt's Third Circle #05	0	100	200	300	398	502	598	700	800	900	1000	1098	1200	1706	3	WT	Germany	0.73	1.26	13.45	16.92	8.36	10.08	Tuning and Temperament[i]
Malcolm	0	112	204	316	386	498	590	702	814	884	996	1088	1200	1721	4	J	England	9.64	11.54	9.64	20.03	10.36	15.58	Tuning and Temperament[i]
Malcolm (just w/mean semitones)	0	105	204	298	386	498	603	702	796	884	989	1088	1200	1721	4	MMT	England	6.82	8.92	10.82	17.71	10.45	14.42	Tuning and Temperament[i]
Neidhardt's Circulating #2	0	96	196	298	394	500	596	698	796	894	1000	1096	1200	1724	4	WT	Germany	3.27	4.00	12.00	14.99	10.55	12.77	Tuning and Temperament[i]
Neidhardt's Circulating #3	0	96	196	298	394	498	596	696	796	894	998	1096	1200	1724	4	WT	Germany	3.82	4.24	12.18	15.30	10.36	12.79	Tuning and Temperament[i]
Rameau's Mod. Meantone	0	87	193	298	386	503	585	697	789	890	1007	1083	1200	1726	4	MMT	France	9.27	11.05	9.09	11.31	16.55	19.91	Tuning and Temperament[i]
Neidhardt's Sample #2	0	90	194	294	386	496	590	698	792	890	994	1088	1200	1732	4	MJ	Germany	8.00	9.12	9.82	14.20	13.45	17.27	Tuning and Temperament[i]
Neidhardt's Sample #3	0	92	196	296	388	498	592	698	794	892	996	1090	1200	1732	4	MJ	Germany	6.18	7.27	10.18	14.17	12.00	15.65	Tuning and Temperament[i]
Bendeler #1	0	90	188	294	392	498	596	694	792	890	996	1094	1200	1739	4	WT	Germany	6.91	7.85	12.00	14.94	12.36	15.80	Tuning and Temperament[i]
Bendeler #2	0	90	196	294	392	498	596	694	792	890	996	1094	1200	1739	4	WT	Germany	6.18	6.99	11.27	14.28	11.64	15.18	Tuning and Temperament[i]
Bendeler #3	0	96	192	294	396	498	594	696	798	894	996	1092	1200	1739	4	WT	Germany	4.91	5.55	12.91	16.53	10.36	13.28	Tuning and Temperament[i]

Euler	0	70	204	274	386	498	590	702	772	884	976	1088	1200	1739	4	J	Russia	15.27	19.12	7.64	18.78	19.64	25.71	Tuning and Temperament[i]
Leven #1	0	112	231	316	404	498	597	702	814	933	996	1129	1200	1743	4	LD	France	13.64	18.80	22.55	30.32	10.73	15.97	Tuning and Temperament[i]
Leven #2	0	112	231	316	404	498	597	702	814	902	996	1095	1200	1743	4	LD	France	8.64	12.79	16.64	23.42	8.27	13.05	Tuning and Temperament[i]
Silbermann (-1/6)	0	89	197	305	394	502	590	698	787	895	1003	1092	1200	1748	4	MT	Germany	6.18	7.52	8.91	11.02	13.82	16.86	Tuning and Temperament[i]
Harrison (-3/10)	0	69	191	314	382	504	573	696	764	887	1009	1078	1200	1749	4	MT	England	17.00	20.86	7.18	8.97	24.64	30.20	Tuning and Temperament[i]
Smith (5/18)	0	72	192	312	384	504	576	696	768	888	1008	1080	1200	1749	4	MT	England	15.27	18.68	6.55	7.95	22.91	28.01	Tuning and Temperament[i]
Gallimard's Mod. Meantone #1	0	84	193	297	386	504	582	696	789	890	1007	1083	1200	1754	4	MMT	France	10.09	11.94	9.36	11.34	17.18	20.79	Tuning and Temperament[i]
Gallimard's Mod. Meantone #2	0	81	193	293	386	504	581	696	785	890	1007	1083	1200	1754	4	MMT	France	11.18	13.08	9.18	11.29	17.73	21.90	Tuning and Temperament[i]
Montvallon	0	92	204	316	386	498	590	702	794	884	996	1088	1200	1758	4	J	France	8.55	10.45	6.00	12.05	14.00	18.41	Tuning and Temperament[i]
Rousseau	0	70	204	316	386	498	568	702	814	884	954	1088	1200	1768	4	J	France	17.09	22.61	11.64	25.19	20.00	27.52	Tuning and Temperament[i]
Marpurg Temperament I	0	105	204	302	400	506	604	702	800	906	1004	1102	1200	1776	4	WT	Poland	3.18	3.96	14.82	18.81	6.64	8.49	Tuning and Temperament[i]
Marpurg's Monochord #1	0	70	204	316	386	498	590	702	772	884	1018	1088	1200	1776	4	J	Poland	13.82	17.27	0.00	0.00	20.00	26.03	Tuning and Temperament[i]
Marpurg's Monochord #2	0	92	204	316	386	498	590	702	794	906	1018	1088	1200	1776	4	J	Poland	8.91	10.86	6.00	12.05	14.00	18.41	Tuning and Temperament[i]
Marpurg's Monochord #3	0	70	204	306	386	498	590	702	772	906	996	1088	1200	1776	4	J	Poland	10.73	14.95	4.91	10.33	15.09	23.38	Tuning and Temperament[i]
Marpurg's Monochord #4	0	70	182	316	386	498	568	702	772	884	1018	1088	1200	1776	4	J	Poland	17.09	20.53	4.00	9.84	24.00	29.52	Tuning and Temperament[i]
Marpurg's Temperament #1	0	102	202	304	400	502	602	704	800	902	1002	1104	1200	1776	4	WT	Poland	2.18	2.68	14.18	17.75	7.27	8.81	Tuning and Temperament[i]
Marpurg's Temperament #2	0	96	194	297	400	496	594	697	800	896	994	1097	1200	1776	4	WT	Poland	3.55	4.28	13.91	17.21	9.55	11.74	Tuning and Temperament[i]
Marpurg's Temperament A	0	102	200	300	402	500	602	700	802	902	1000	1102	1200	1776	4	WT	Poland	1.09	1.55	14.91	18.57	6.55	7.95	Tuning and Temperament[i]
Marpurg's Temperament B	0	98	198	298	400	500	600	698	798	898	1000	1100	1200	1776	4	WT	Poland	1.09	1.55	13.82	16.73	8.36	10.20	Tuning and Temperament[i]
Marpurg's Temperament C	0	98	200	300	400	498	600	700	800	898	1000	1100	1200	1776	4	WT	Poland	0.55	1.10	13.27	16.75	7.82	9.78	Tuning and Temperament[i]
Marpurg's Temperament D	0	98	198	300	398	498	600	698	798	900	998	1098	1200	1776	4	WT	Poland	1.45	1.79	13.45	16.64	8.36	10.47	Tuning and Temperament[i]
Marpurg's Temperament E	0	100	202	302	402	502	602	700	800	900	1000	1100	1200	1776	4	WT	Poland	0.91	1.41	14.00	17.40	7.45	9.10	Tuning and Temperament[i]
Marpurg's Temperament G (1/5 comma)	0	100	199	299	398	498	602	702	802	901	1001	1100	1200	1776	4	WT	Poland	1.27	1.55	13.82	17.67	7.09	8.90	Tuning and Temperament[i]

Mercadier (1/12,1/6)	0	94	197	296	394	500	594	698	794	895	998	1094	1200	1777	4	WT	France	4.18	4.88	11.64	14.72	10.73	13.63	Tuning and Temperament[i]
Kirnberger #1	0	90	204	294	386	498	590	702	792	895	996	1088	1200	1779	4	WT	Germany	7.00	8.40	8.64	13.74	11.36	16.51	http://www.terryblackburn.us/music/temperament/#project
Kirnberger #3	0	90	193	294	386	498	590	697	792	890	996	1088	1200	1779	4	WT	Germany	7.72	8.90	9.73	13.97	13.23	17.19	http://www.terryblackburn.us/music/temperament/#project
Vallotti Circulating	0	94	196	298	392	502	592	698	796	894	1000	1090	1200	1781	4	WT	Italy	4.73	5.87	10.91	14.28	12.00	14.59	http://music.cwru.edu/duffin/Vallotti/T1/page2.html http://mmd.foxtail.com/Tech/jorgensen.html
von Wiese #1	0	90	204	294	408	498	600	702	792	906	996	1110	1200	1793	4	WT	Germany	5.45	6.63	14.55	18.22	5.45	11.38	Tuning and Temperament[i]
von Wiese #2	0	90	204	294	386	498	590	702	792	895	996	1088	1200	1793	4	WT	Germany	7.00	8.40	8.64	13.74	11.36	16.51	Tuning and Temperament[i]
von Wiese #3	0	102	204	306	408	498	600	702	804	906	996	1110	1200	1793	4	WT	Germany	4.36	5.44	15.64	20.45	4.36	7.59	Tuning and Temperament[i]
Young Circulating #1	0	94	196	298	392	500	592	698	796	894	1000	1092	1200	1800	4	WT	England	4.36	5.51	10.91	14.28	11.64	14.28	Tuning and Temperament[i]
Young Circulating #2	0	90	196	294	392	498	588	698	792	894	996	1090	1200	1800	4	WT	England	6.55	7.64	10.55	14.11	12.00	16.15	http://music.cwru.edu/duffin/Vallotti/T1/page2.html
Stanhope (1/3 comma)	0	91	197	295	386	498	589	702	793	892	996	1088	1200	1806	4	WT	England	7.00	8.44	9.18	13.86	12.27	16.74	Tuning and Temperament[i]

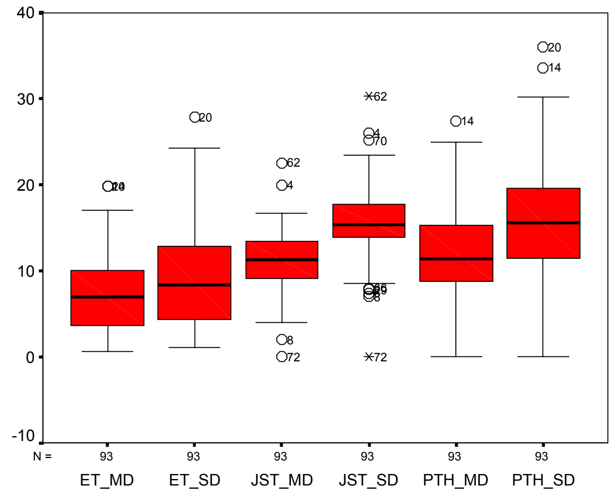
P=Pythagorean
MT=Mean Tone
MMT=Modified Mean Tone
J=Just Intonation
MJ=Modified Just Intonation
WT=Well Temperament
LD=Linear Divisions (not common)

ET_MD =Equal Temperament Standard, Mean Deviation
ET_SD =Equal Temperament Standard, Standard Deviation
JST_MD=Just Intonation Standard, Mean Deviation
JST_SD =Just Intonation Standard, Standard Deviation
PTH_MD=Pythagorean Standard, Mean Deviation
PTH_SD=Pythagorean Standard, Standard Deviation

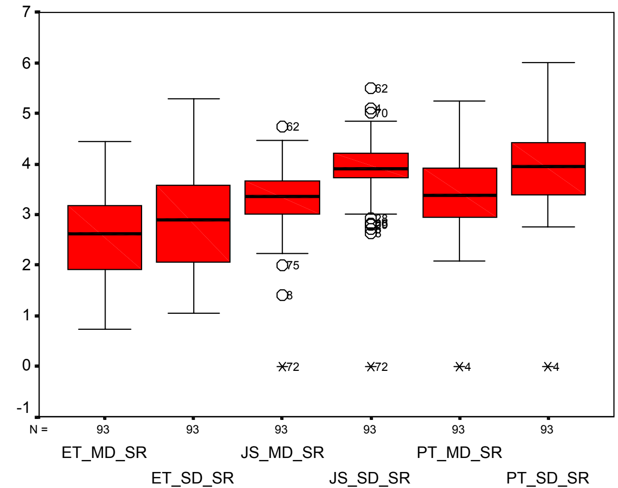
i. Barbour, J. Murry. *Tuning and Tmperament – A Historical Survey,* Michigan State College Press, East Lansing (1951)

APPENDIX B -- SPREAD & LINEAR REGRESSIONS ON ORIGINAL DATA

Spread of Data (all 93)

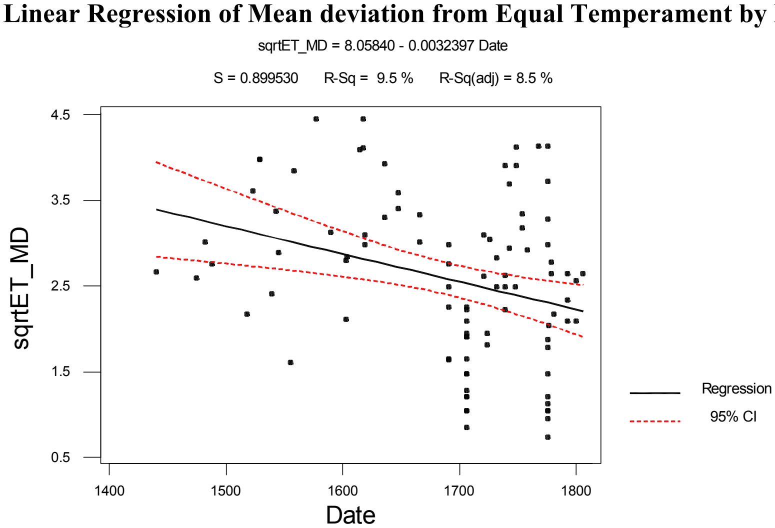


Spread of Data with Square Root Transformation (all 93)



ET_MD =Equal Temperament Standard, Mean Deviation
ET_SD =Equal Temperament Standard, Standard Deviation
JST_MD=Just Intonation Standard, Mean Deviation
JST_SD =Just Intonation Standard, Standard Deviation
PT_MD=Pythagorean Standard, Mean Deviation
PT_SD=Pythagorean Standard, Standard Deviation
SR after a variable name indicates square root transformation

Linear Regression of Mean deviation from Equal Temperament by Date



Regression Analysis: sqrtET_MD versus Date, And Countries with Residual Graphs

```
The regression equation is
sqrtET_MD = 5.91 - 0.00161 Date - 1.10 Poland - 0.981 Germany

Predictor         Coef    SE Coef        T      P
Constant         5.910      1.639     3.60  0.001
Date        -0.0016092  0.0009841    -1.64  0.106
Poland         -1.1008     0.2729    -4.03  0.000
Germany        -0.9810     0.1751    -5.60  0.000

S = 0.7701      R-Sq = 35.2%     R-Sq(adj) = 33.0%
```

Analysis of Variance

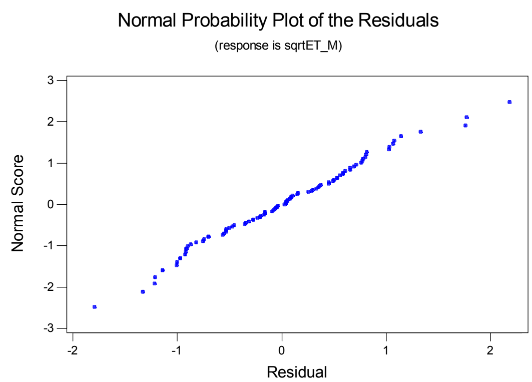
Source	DF	SS	MS	F	P
Regression	3	28.6155	9.5385	16.08	0.000
Residual Error	89	52.7814	0.5930		
Total	92	81.3969			

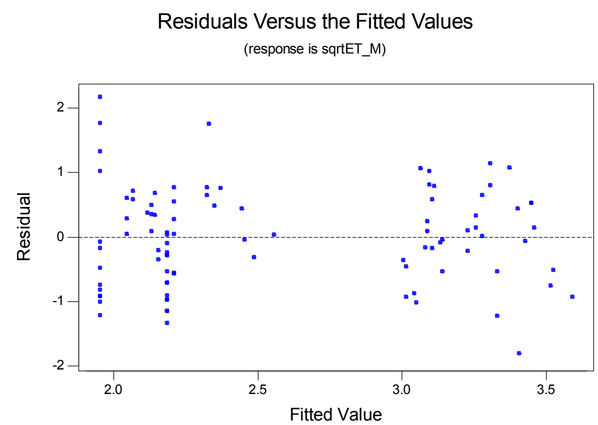
Source	DF	Seq SS
Date	1	7.7638
Poland	1	2.2260
Germany	1	18.6257

Unusual Observations

Obs	Date	sqrtET_M	Fit	SE Fit	Residual	St Resid
12	1555	1.6100	3.4072	0.1640	-1.7972	-2.39R
19	1615	4.0900	2.3297	0.1388	1.7603	2.32R
72	1776	3.7200	1.9508	0.2136	1.7692	2.39R
75	1776	4.1300	1.9508	0.2136	2.1792	2.95R

R denotes an observation with a large standardized residual





All Regressions Tried on Original Data Set

Regression Analysis: sqrtET_MD versus Date

The regression equation is
sqrtET_MD = 8.05840 - 0.0032397 Date

S = 0.899530 R-Sq = 9.5 % R-Sq(adj) = 8.5 %

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	1	7.7638	7.76384	9.59500	0.003
Error	91	73.6331	0.80915		
Total	92	81.3969			

Regression Analysis: sqrtET_SD versus Date

The regression equation is
sqrtET_SD = 8.88756 - 0.0035527 Date

S = 0.996317 R-Sq = 9.4 % R-Sq(adj) = 8.4 %

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	1	9.3366	9.33661	9.40576	0.003
Error	91	90.3309	0.99265		
Total	92	99.6675			

Regression Analysis: sqrtJST_MD versus Date

The regression equation is
sqrtJST_MD = 3.57581 - 0.0001798 Date

S = 0.623986 R-Sq = 0.1 % R-Sq(adj) = 0.0 %

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	1	0.0239	0.023918	6.14E-02	0.805
Error	91	35.4317	0.389359		
Total	92	35.4556			

Regression Analysis: sqrtPTH_MD versus Date

The regression equation is
sqrtPTH_MD = 3.65839 - 0.0001224 Date

S = 0.807874 R-Sq = 0.0 % R-Sq(adj) = 0.0 %

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	1	0.0111	0.011079	1.70E-02	0.897
Error	91	59.3921	0.652661		
Total	92	59.4032			

Regression Analysis: sqrtPTH_SD versus Date

```
The regression equation is
sqrtPTH_SD = 5.15643 - 0.0007037 Date

S = 0.882887      R-Sq = 0.5 %      R-Sq(adj) = 0.0 %

Analysis of Variance

Source            DF        SS          MS        F       P
Regression         1    0.3663    0.366273 0.469889  0.495
Error             91   70.9335    0.779489
Total             92   71.2998
```

Regression Analysis: sqrtPTH_SD versus Date

```
The regression equation is
sqrtPTH_SD = 5.15643 - 0.0007037 Date

S = 0.882887      R-Sq = 0.5 %      R-Sq(adj) = 0.0 %

Analysis of Variance

Source            DF        SS          MS        F       P
Regression         1    0.3663    0.366273 0.469889  0.495
Error             91   70.9335    0.779489
Total             92   71.2998
```

Regression Analysis: sqrtJST_MD versus Date

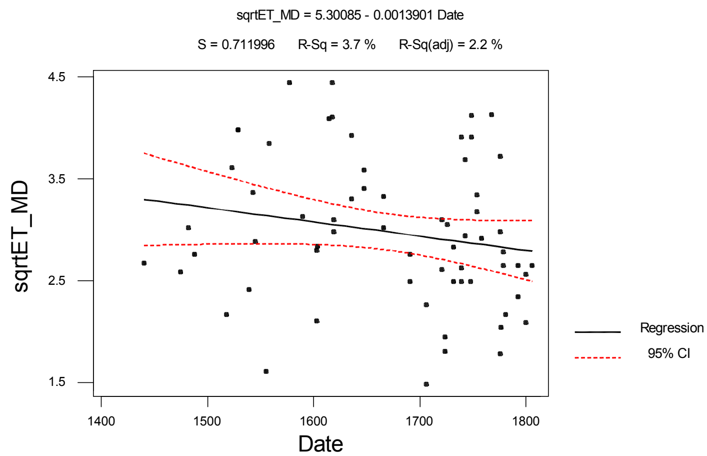
```
The regression equation is
sqrtJST_MD = 3.57581 - 0.0001798 Date

S = 0.623986      R-Sq = 0.1 %      R-Sq(adj) = 0.0 %

Analysis of Variance

Source            DF        SS          MS        F       P
Regression         1    0.0239    0.023918 6.14E-02  0.805
Error             91   35.4317    0.389359
Total             92   35.4556
```

Regression Analysis: sqrtJST_SD versus Date

```
The regression equation is
sqrtJST_SD = 5.46406 - 0.0009327 Date

S = 0.652843      R-Sq = 1.6 %      R-Sq(adj) = 0.6 %

Analysis of Variance

Source            DF        SS          MS        F       P
Regression         1    0.6435    0.643524  1.50990  0.222
Error             91   38.7845    0.426203
Total             92   39.4280
```

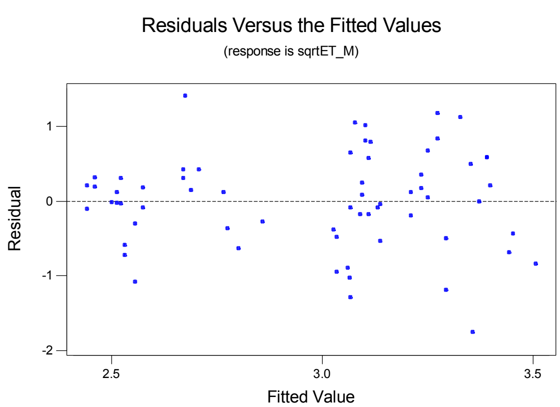
APPENDIX C -- REGRESSIONS ON THE MODIFIED DATA (TWO TUNINGS PER PUBLICATION)

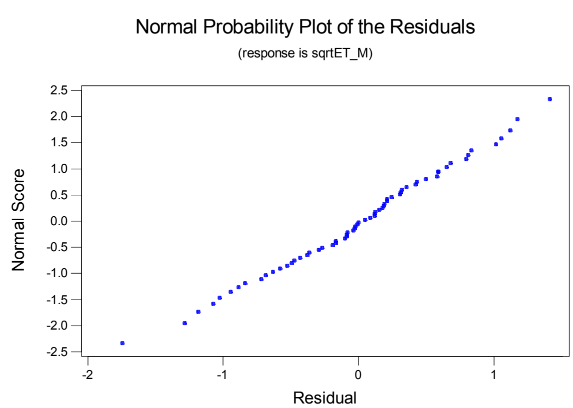
Regression of Mean Deviation (with a square root transformation) from Equal Temperament by Date



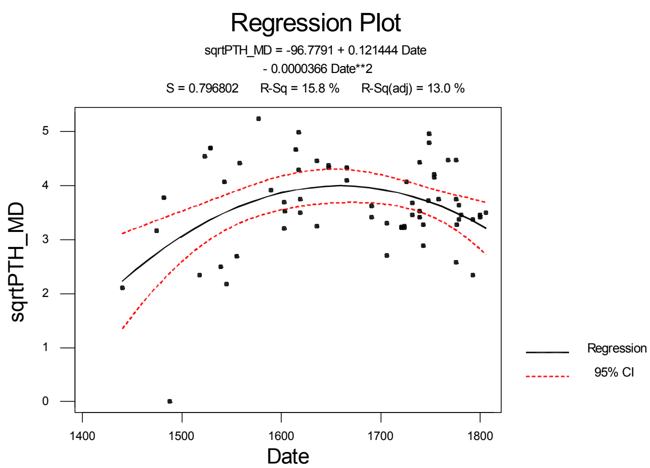
Regression Analysis: sqrtET_MD versus Date and Country of Publication (filtered by stepwise)

```
sqrtET_MD = 5.40 - 0.00131 Date - 0.603 Germany

Predictor         Coef     SE Coef       T      P
Constant         5.396       1.379    3.91  0.000
Date        -0.0013119   0.0008239   -1.59  0.116
Germany        -0.6030      0.1686   -3.58  0.001

S = 0.6526      R-Sq = 20.4%     R-Sq(adj) = 17.8%

Analysis of Variance
Source            DF        SS        MS       F      P
Regression         2    6.6643    3.3322    7.82  0.001
Residual Error    61   25.9790    0.4259
Total             63   32.6433

Source       DF      Seq SS
Date          1      1.2131
Germany       1      5.4512

Unusual Observations
Obs        Date    sqrtET_M       Fit     SE Fit    Residual    St Resid
 12        1555      1.6100    3.3563     0.1399     -1.7463      -2.74R
 19        1615      4.0900    2.6745     0.1421      1.4155       2.22R
 53        1776      1.7800    3.0664     0.1354     -1.2864      -2.02R
```



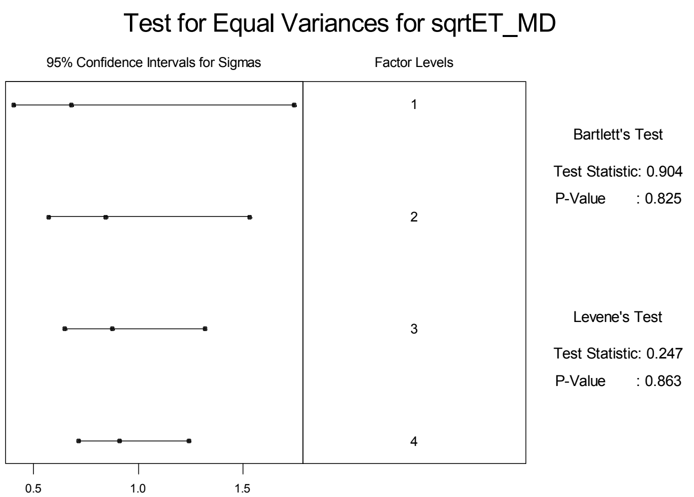


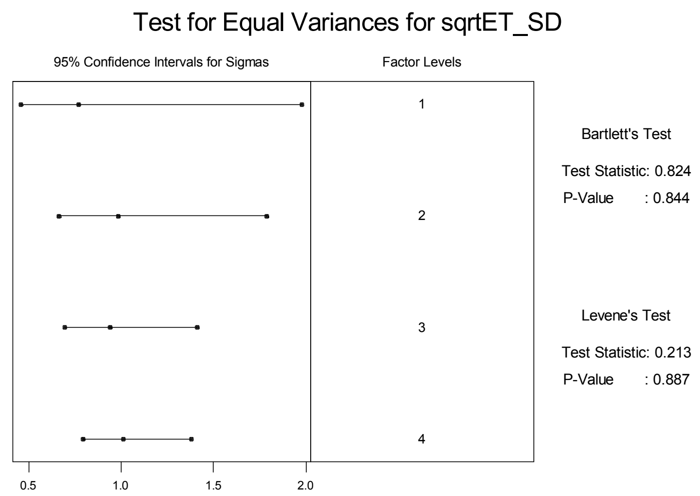
Various Curve Fitting Model Attempts

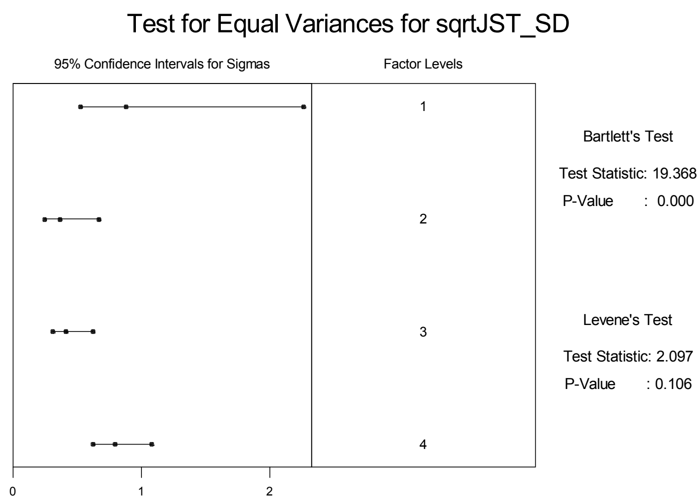


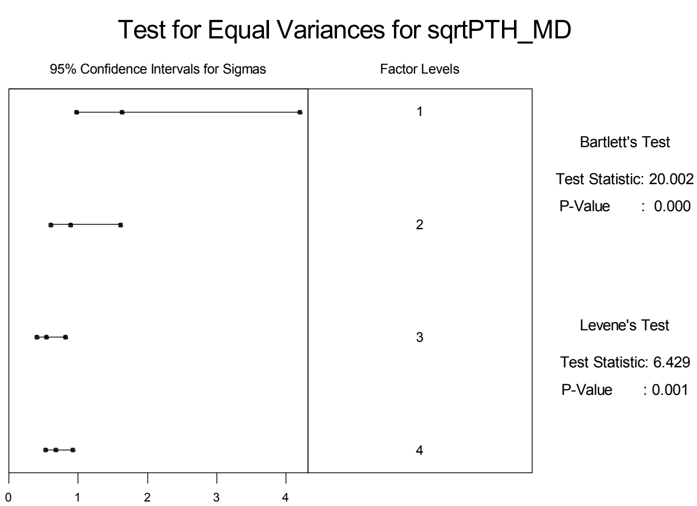
ET_MD =Equal Temperament Standard, Mean Deviation
ET_SD =Equal Temperament Standard, Standard Deviation
JST_MD=Just Intonation Standard, Mean Deviation
JST_SD =Just Intonation Standard, Standard Deviation
PTH_MD=Pythagorean Standard, Mean Deviation
PTH_SD=Pythagorean Standard, Standard Deviation
sqrt before a variable name indicates square root transformation

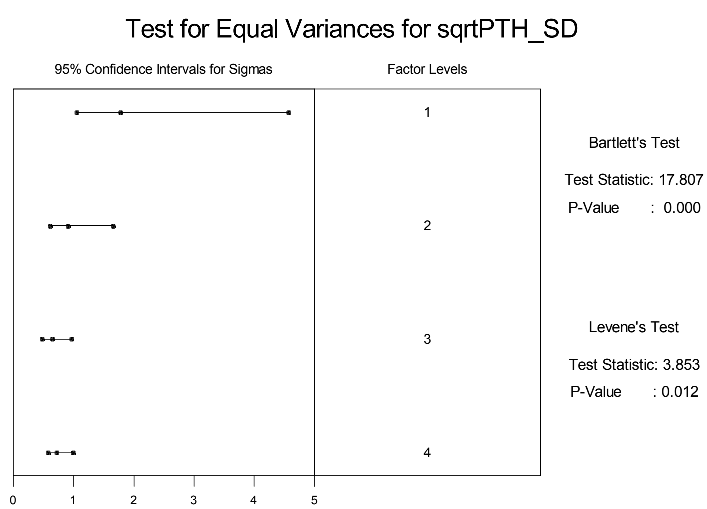
APPENDIX D -- VARIANCE TESTS ON THE ORIGINAL DATA











ET_MD =Equal Temperament Standard, Mean Deviation
ET_SD =Equal Temperament Standard, Standard Deviation
JST_MD=Just Intonation Standard, Mean Deviation
JST_SD =Just Intonation Standard, Standard Deviation
PTH_MD=Pythagorean Standard, Mean Deviation
PTH_SD=Pythagorean Standard, Standard Deviation
sqrt before a variable name indicates square root transformation

ANOVA's and Mood's median tests performed on original data set.

Analysis of Variance for sqrtET_M

```
Source     DF        SS        MS        F        P
Quarters    3    13.305     4.435     5.80    0.001
Error      89    68.092     0.765
Total      92    81.397
                                   Individual 95% CIs For Mean
Level       N      Mean     StDev  ----------+---------+---------+------
1           8    3.0975    0.6804             (----------*---------)
2          15    3.2127    0.8445                  (-------*------)
3          27    2.1459    0.8771  (-----*----)
4          43    2.5393    0.9111          (---*----)
                                   ----------+---------+---------+------
Pooled StDev =   0.8747                   2.40      3.00      3.60
```

Analysis of Variance for sqrtET_S

```
Source     DF        SS        MS        F        P
Quarters    3    16.005     5.335     5.68    0.001
Error      89    83.662     0.940
Total      92    99.668
                                   Individual 95% CIs For Mean
Level       N      Mean     StDev  ------+---------+---------+---------+
1           8    3.4325    0.7710              (----------*-----------)
2          15    3.5913    0.9847                    (-------*-------)
3          27    2.4148    0.9394  (-----*-----)
4          43    2.8249    1.0116          (----*----)
                                   ------+---------+---------+---------+
Pooled StDev =   0.9696               2.40      3.00      3.60      4.20
```

Mood Median Test: sqrtJST_MD versus Quarters

```
Chi-Square = 2.95   DF = 3   P = 0.399
                                        Individual 95.0% CIs
Quarters    N<=     N>   Median    Q3-Q1  ---+---------+---------+---------+---
1             5      3     3.15     1.49   (--------------+---------------)
2             8      7     3.23     0.54                  (-+-------)
3            10     17     3.46     0.45                      (-+---)
4            24     19     3.30     0.74                 (---+--)
                                        ---+---------+---------+---------+---
                                        2.50      3.00      3.50      4.00
Overall median = 3.36
```

Mood Median Test: sqrtJST_SD versus Quarters

```
Chi-Square = 6.86   DF = 3   P = 0.076
                                        Individual 95.0% CIs
Quarters    N<=     N>   Median    Q3-Q1  -----+---------+---------+---------+-
1             4      4     3.93     1.69  (-----------------+------------)
2             3     12     4.21     0.30                         (-+-)
3            15     12     3.89     0.33                     (+--)
4            25     18     3.84     0.74                    (+---)
                                        -----+---------+---------+---------+-
                                          3.00      3.60      4.20      4.80
Overall median = 3.91
```

Mood Median Test: sqrtPTH_MD versus Quarters

```
Chi-Square = 6.28   DF = 3   P = 0.099
                                        Individual 95.0% CIs
Quarters    N<=     N>   Median    Q3-Q1  ------+---------+---------+---------+
1             4      4     3.47     2.49   (----------------+---------------)
2             4     11     3.74     1.21                 (-----+-------)
3            18      9     3.25     0.67             (---+-)
4            21     22     3.41     0.85                (--+)
                                        ------+---------+---------+---------+
                                           2.40      3.20      4.00      4.80
Overall median = 3.37
```

Mood Median Test: sqrtPTH_SD versus Quarters

```
Chi-Square = 7.95   DF = 3   P = 0.047
                                        Individual 95.0% CIs
Quarters    N<=     N>   Median    Q3-Q1  ------+---------+---------+---------+
1             4      4     4.13     2.28  (----------------+-------------)
2             4     11     4.29     1.70              (------+---------)
3            19      8     3.59     0.84         (--+---)
4            20     23     3.96     0.86             (--+-)
                                        ------+---------+---------+---------+
                                           3.20      4.00      4.80      5.60
Overall median = 3.94
```

APPENDIX E – AVERAGES FROM EACH PUBLICATION

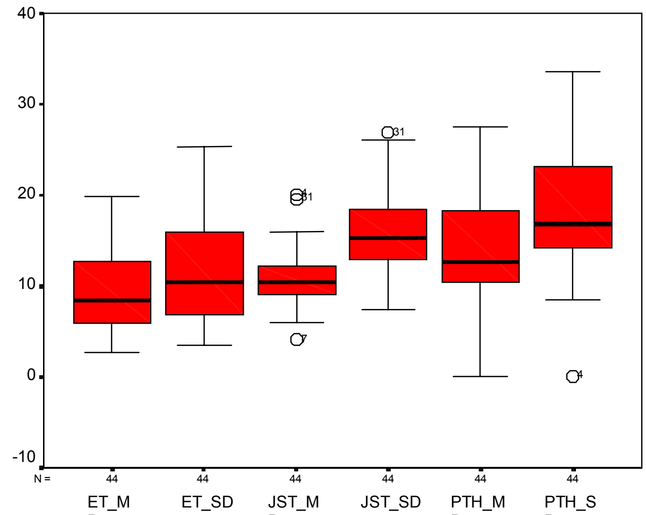
	Date	Quarters	Type	Country	ET_MD	ET_SD	JST_MD	JST_SD	PTH_MD	PTH_SD
van Zwolle (transposed Pythagorean)	1440	1	P	France	7.11	8.61	15.91	21.69	4.44	10.62
Erlangen (anonymous, transposed Pythagorean	1475	1	P	Germany	6.73	8.07	10.00	15.56	10.00	15.56
Ramis (transposed Pythagorean)	1482	1	P	Spain	9.09	10.99	9.82	15.28	14.18	18.65
Pythagorean (published by Hugo de Reutlingen)	1488	1	P	France	7.64	9.34	20.00	26.03	0.00	0.00
Grammateus'	1518	1	MMT	Germany	4.73	5.87	14.55	19.49	5.45	8.49
Meantone (-1/4 Aron)	1523	1	MT	Italy	13.00	15.96	6.09	7.40	20.64	25.30
Fogliano 1-2	1529	1	J	Italy	15.82	19.77	4.00	9.50	22.00	28.68
Agricola	1539	2	J	Germany	5.82	6.75	13.82	18.18	6.18	12.07
Ganassi (just w/mean semitones)	1543	2	MMT	Italy	11.36	13.53	12.27	19.06	16.45	19.62
Agricola	1545	2	WT	Germany	8.36	9.92	15.27	22.32	4.73	9.21
Bermudo's Vihuela	1555	2	WT	Spain	2.59	3.62	14.10	17.70	7.25	10.38
Zarlino (-2/7)	1558	2	MT	Italy	14.82	17.76	10.45	16.58	19.55	24.77
Salinas (-1/3)	1577	2	MT	Spain	19.82	24.29	8.91	11.30	27.45	33.63
Schneegass (geometric)	1590	2	MT	Germany	9.82	11.76	9.82	14.69	15.27	19.00
Artusi 1-2	1603	2	MMT	Italy	6.14	7.55	13.32	17.26	11.95	14.54
Reinhard (just w/mean semitones)	1604	2	MMT	Germany	8.09	10.41	10.27	17.36	12.45	16.59
De Caus	1615	2	J	Germany	16.73	20.80	9.82	17.93	21.82	28.54
Colonna 1-2	1618	2	MJ	Italy	18.36	25.31	11.45	20.49	21.64	31.93
Kepler 1-2	1619	2	J	Germany	9.27	11.22	6.91	14.29	13.09	17.73
Mersenne 1-2	1636	3	J	France	13.18	16.43	10.64	19.33	15.18	21.64
Mersenne Improved Meantone 1-2	1648	3	MMT	France	12.27	15.74	8.36	11.36	18.91	24.63
Rossi 1/5 and 2/9	1666	3	MT	Italy	10.09	12.40	7.00	8.17	17.73	21.73
Werckmeister 1-6	1691	3	WT	Germany	5.54	6.74	10.98	13.92	11.73	15.14
Neidhardt's 1st Pub	1706	3	WT	Germany	2.67	3.39	12.92	16.33	9.09	11.42
Malcolm 1-2	1721	4	J	England	8.23	10.23	10.23	18.87	10.41	15.00
Neidhardt's 2nd Pub	1724	4	WT	Germany	3.55	4.12	12.09	15.15	10.45	12.78
Rameau's Modified Meantone	1726	4	MMT	France	9.27	11.05	9.09	11.31	16.55	19.91

Neidhardt's 3rd Pub	1732	4	MJ	Germany	7.09	8.19	10.00	14.18	12.73	16.46
Bendeler 1-3	1739	4	WT	Germany	6.00	6.79	12.06	15.25	11.45	14.75
Euler	1739	4	J	Russia	15.27	19.12	7.64	18.78	19.64	25.71
Leven 1-2	1743	4	LD	France	11.14	15.80	19.59	26.87	9.50	14.51
Silbermann (-1/6)	1748	4	MT	Germany	6.18	7.52	8.91	11.02	13.82	16.86
Harrison (-3/10)	1749	4	MT	England	17.00	20.86	7.18	8.97	24.64	30.20
Smith (5/18)	1749	4	MT	England	15.27	18.68	6.55	7.95	22.91	28.01
Gallimard's Modified Meantone 1-2	1754	4	MMT	France	10.64	12.51	9.27	11.31	17.45	21.34
Montvallon	1758	4	J	France	8.55	10.45	6.00	12.05	14.00	18.41
Rousseau	1768	4	J	France	17.09	22.61	11.64	25.19	20.00	27.52
Marpurg's Pub	1776	4	WT	Poland	5.06	6.42	10.85	14.60	10.94	14.06
Mercadier (1/12,1/6)	1777	4	WT	France	4.18	4.88	11.64	14.72	10.73	13.63
Kirnberger 1&3	1779	4	WT	Germany	7.36	8.65	9.18	13.86	12.30	16.85
Vallotti Circulating	1781	4	WT	Italy	4.73	5.87	10.91	14.28	12.00	14.59
von Wiese 1-3	1793	4	WT	Germany	5.61	6.82	12.94	17.47	7.06	11.83
Young 1-2	1800	4	WT	England	5.45	6.58	10.73	14.20	11.82	15.22
Stanhope (1/3 comma)	1806	4	WT	England	7.00	8.44	9.18	13.86	12.27	16.74

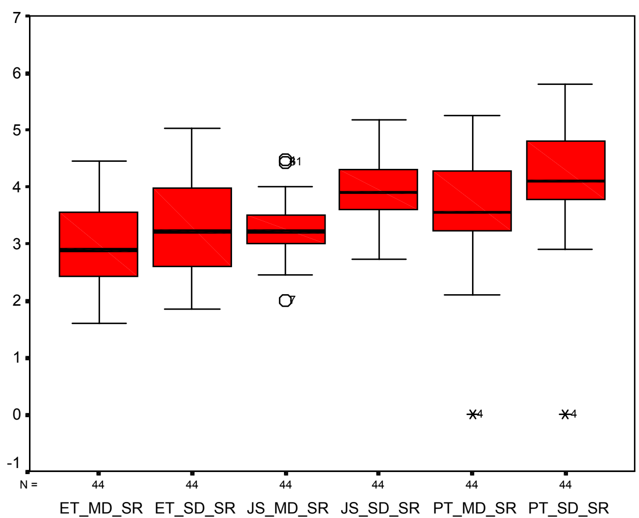
ET_MD =Equal Temperament Standard, Mean Deviation
ET_SD =Equal Temperament Standard, Standard Deviation
JST_MD=Just Intonation Standard, Mean Deviation
JST_SD =Just Intonation Standard, Standard Deviation
PTH_MD=Pythagorean Standard, Mean Deviation
PTH_SD=Pythagorean Standard, Standard Deviation

APPENDIX F – SPREAD OF AVERAGED DATA

Spread of Averaged Data

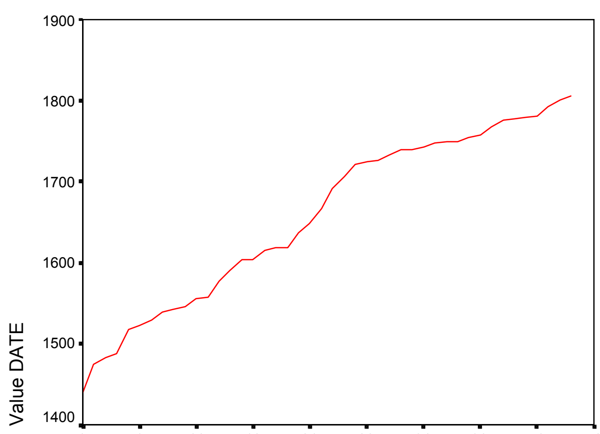


Spread of Transformed (square root) Averaged Data



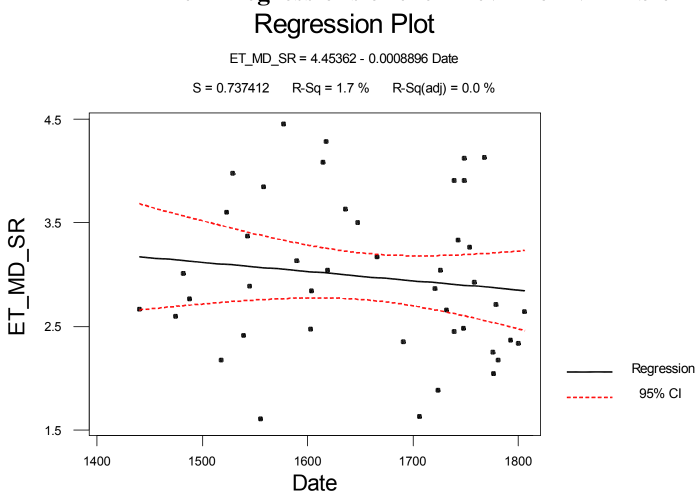
ET_MD =Equal Temperament Standard, Mean Deviation
ET_SD =Equal Temperament Standard, Standard Deviation
JST_MD=Just Intonation Standard, Mean Deviation
JST_SD =Just Intonation Standard, Standard Deviation
PT_MD=Pythagorean Standard, Mean Deviation
PT_SD=Pythagorean Standard, Standard Deviation
SR after a variable name indicates a square root transformation

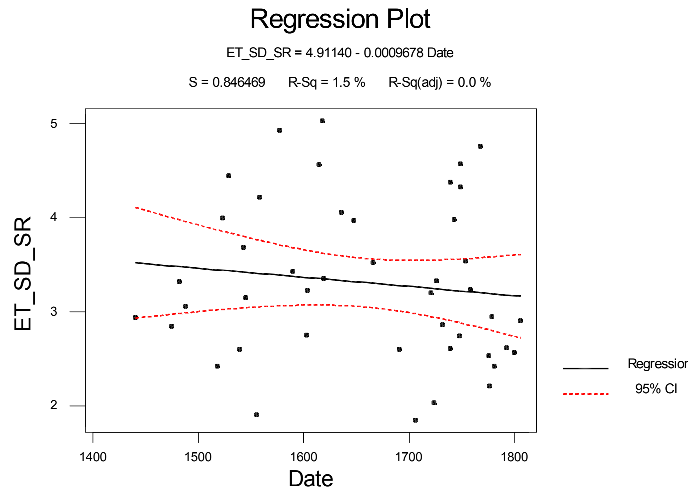
Time Series Plot of Dates

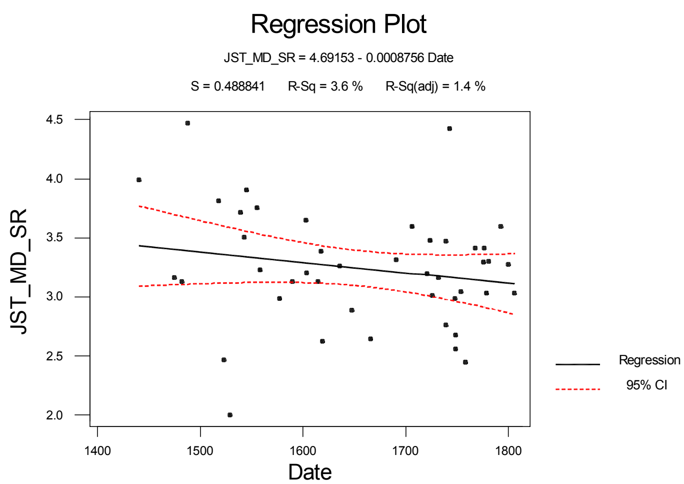


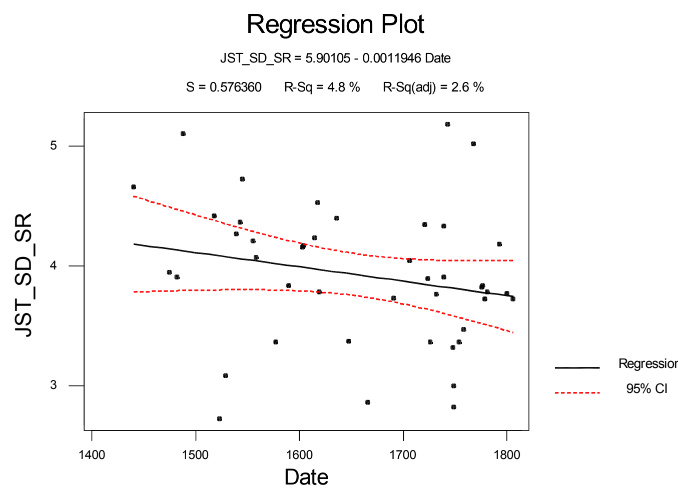
APPENDIX G -- REGRESSION ANALYSIS OF AVERAGED DATA

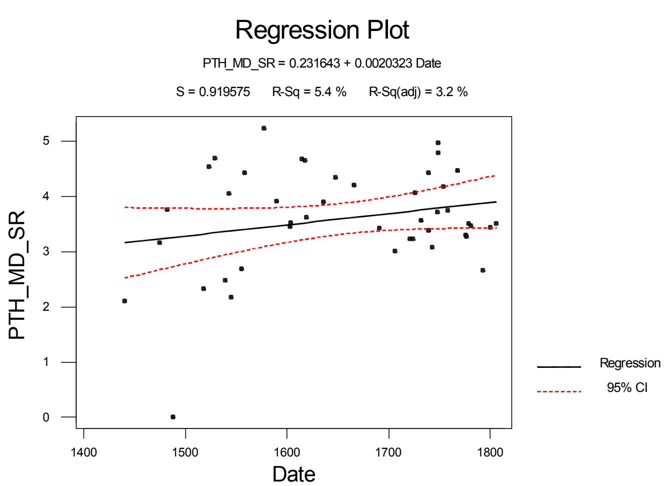
Linear Regressions of each Deviation Variable











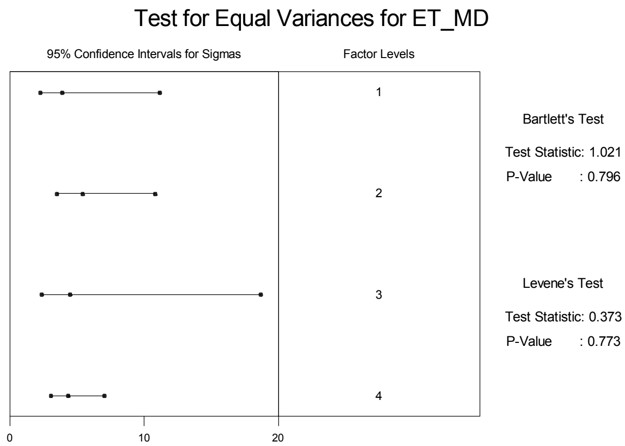
Regression results for best combination of variables (by stepwise method)

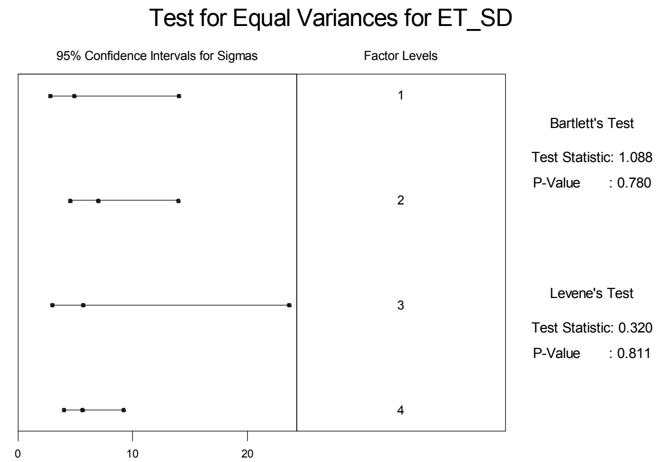
```
The regression equation is
Date = 1696 - 98.2 ET_MD_SR + 90.1 PTH_MD_SR - 140 Italy - 212 Spain
           - 84.0 Germany

Predictor        Coef     SE Coef          T        P
Constant      1696.15       58.83      28.83    0.000
ET_MD_SR       -98.17       22.35      -4.39    0.000
PTH_MD_S        90.11       17.19       5.24    0.000
Italy         -140.46       33.25      -4.22    0.000
Spain         -212.47       47.65      -4.46    0.000
Germany        -84.03       27.81      -3.02    0.004

S = 75.17       R-Sq = 56.6%     R-Sq(adj) = 50.9%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         5      280405       56081      9.92    0.000
Residual Error    38      214738        5651
Total             43      495143

Source       DF      Seq SS
ET_MD_SR      1        8353
PTH_MD_S      1       94693
Italy         1       45481
Spain         1       80270
Germany       1       51608

Unusual Observations
Obs   ET_MD_SR       Date         Fit      SE Fit    Residual    St Resid
  1       2.67     1440.0      1624.2        27.2      -184.2      -2.63R
  2       2.59     1475.0      1642.4        18.9      -167.4      -2.30R
  4       2.76     1488.0      1424.8        59.3        63.2       1.37 X
 11       1.61     1555.0      1568.3        49.7       -13.3      -0.24 X
 13       4.45     1577.0      1518.7        49.8        58.3       1.04 X
 42       2.37     1793.0      1619.0        20.5       174.0       2.41R

R denotes an observation with a large standardized residual
X denotes an observation whose X value gives it large influence.
```

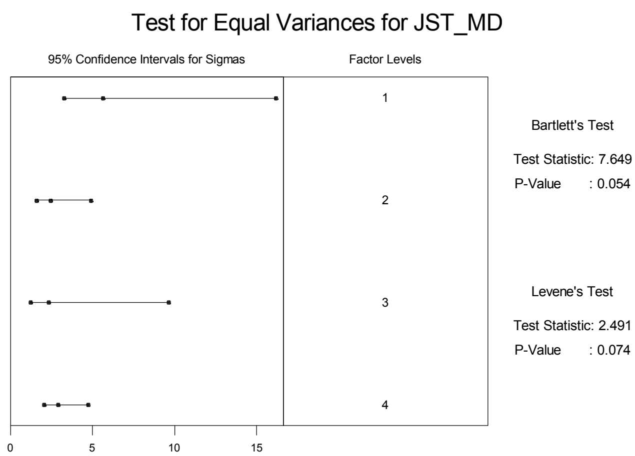
ET_MD =Equal Temperament Standard, Mean Deviation
ET_SD =Equal Temperament Standard, Standard Deviation
JST_MD=Just Intonation Standard, Mean Deviation
JST_SD =Just Intonation Standard, Standard Deviation
PTH_MD=Pythagorean Standard, Mean Deviation
PTH_SD=Pythagorean Standard, Standard Deviation
SR after a variable name indicates a square root transformation

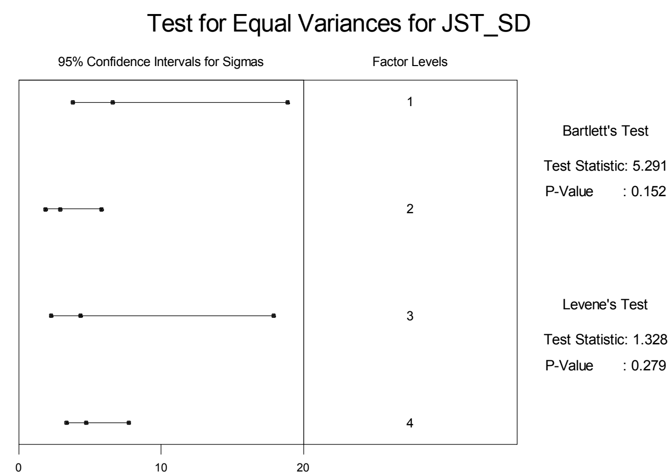
APPENDIX H -- VARIANCE TESTS ON AVERAGED DATA

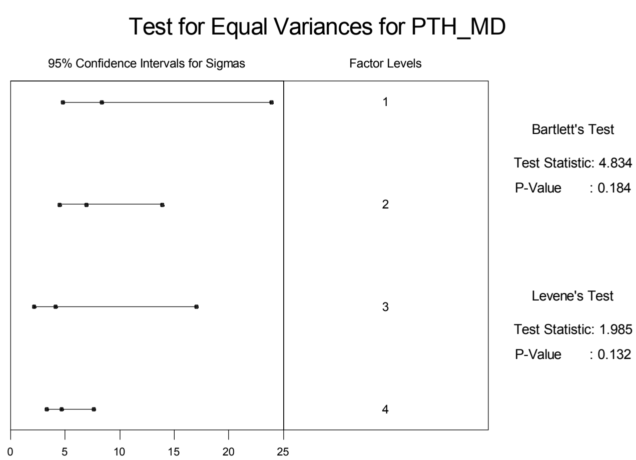
Variances Tests on Averaged Data

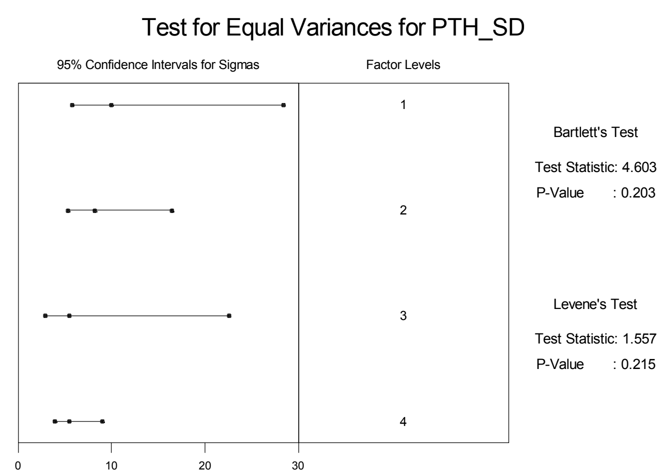




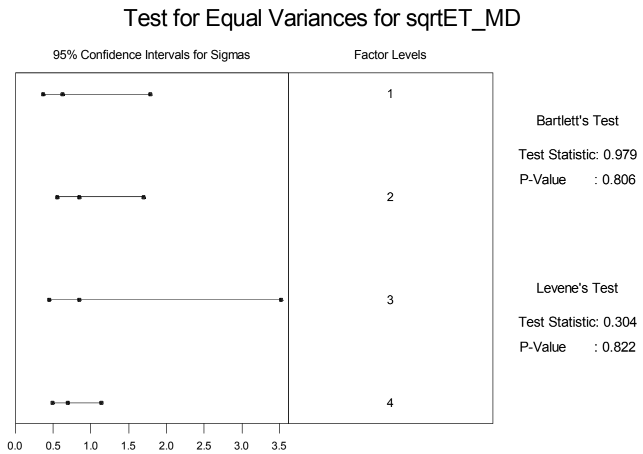


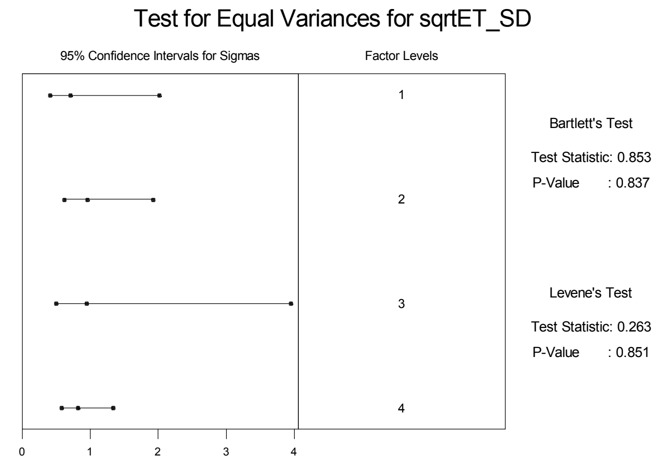


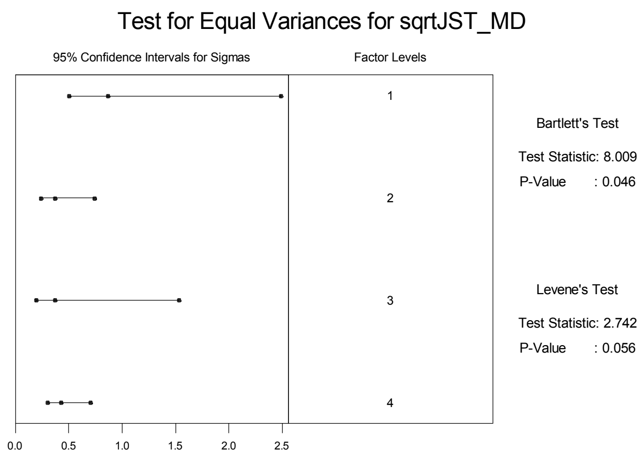


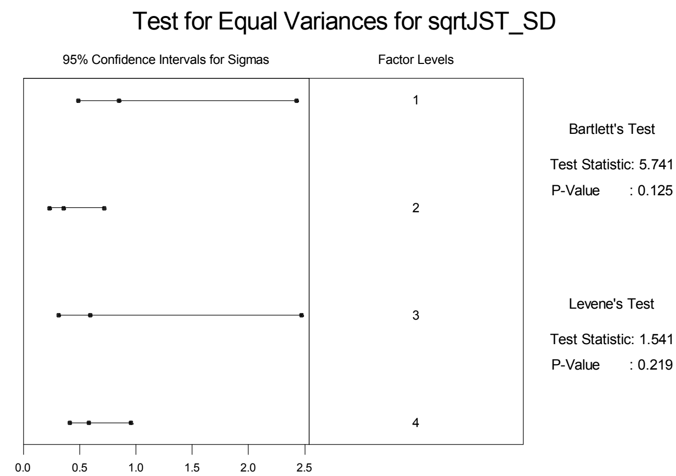


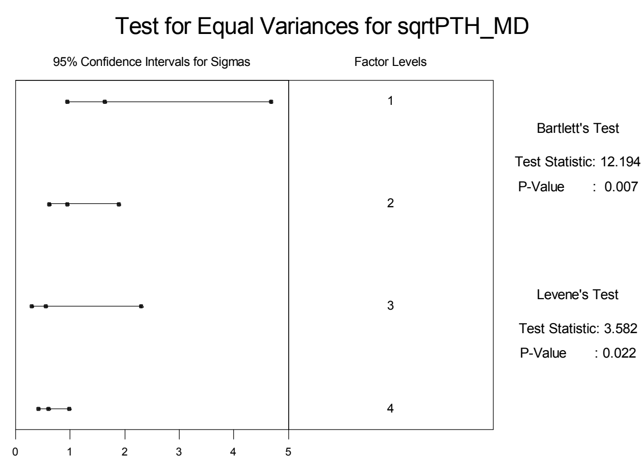
Variances Tests on Transformed Averaged Data (by square root)

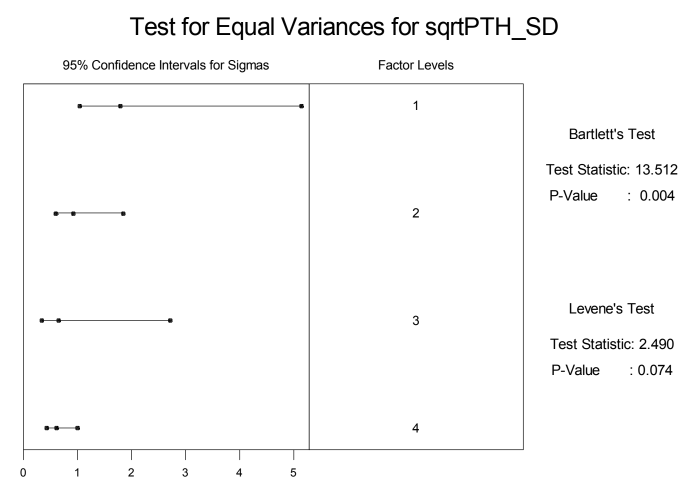












ET_MD =Equal Temperament Standard, Mean Deviation
ET_SD =Equal Temperament Standard, Standard Deviation
JST_MD=Just Intonation Standard, Mean Deviation
JST_SD =Just Intonation Standard, Standard Deviation
PTH_MD=Pythagorean Standard, Mean Deviation
PTH_SD=Pythagorean Standard, Standard Deviation
sqrt before a variable name indicates a square root transformation

All ANOVA's tried on averaged data

One-way ANOVA: sqrtET_MD versus Quarters

```
Analysis of Variance for sqrtET_M
Source     DF        SS        MS        F        P
Quarters    3     0.899     0.300     0.54    0.661
Error      40    22.380     0.559
Total      43    23.278
                                   Individual 95% CIs For Mean
                                   Based on Pooled StDev
Level       N      Mean     StDev  ------+---------+---------+---------+
1           7    2.9714    0.6259        (-------------*--------------)
2          12    3.2050    0.8484                 (-----------*----------)
3           5    2.8580    0.8482   (----------------*----------------)
4          20    2.8760    0.6971           (---------*--------)
                                   ------+---------+---------+---------+
Pooled StDev =   0.7480                 2.40      2.80      3.20      3.60
```

One-way ANOVA: sqrtET_SD versus Quarters

```
Analysis of Variance for sqrtET_S
Source     DF        SS        MS        F        P
Quarters    3     1.184     0.395     0.54    0.659
Error      40    29.364     0.734
Total      43    30.548
                                   Individual 95% CIs For Mean
                                   Based on Pooled StDev
Level       N      Mean     StDev  --+---------+---------+---------+----
1           7    3.2871    0.7045       (------------*------------)
2          12    3.5683    0.9604               (---------*---------)
3           5    3.1960    0.9523  (---------------*---------------)
4          20    3.1845    0.8148          (-------*------)
                                   --+---------+---------+---------+----
Pooled StDev =   0.8568            2.50      3.00      3.50      4.00
```

Mood Median Test: JST_MD versus Quarters

```
Mood median test for JST_MD

Chi-Square = 0.88   DF = 3   P = 0.831

                                        Individual 95.0% CIs
Quarters   N<=     N>   Median    Q3-Q1 -----+---------+---------+---------+-
1            4      3     10.0      9.8  (------------+-------------------)
2            5      7     11.0      3.9             (--+--------)
3            2      3     10.6      4.3      (---------+------)
4           11      9     10.1      2.7           (--+---)
                                        -----+---------+---------+---------+-
                                           7.0      10.5      14.0      17.5
Overall median = 10.4
* NOTE * Levels with < 6 observations have confidence < 95.0%
```

One-way ANOVA: sqrtJST_SD versus Quarters

```
Analysis of Variance for sqrtJST_
Source     DF        SS        MS        F        P
Quarters    3     1.053     0.351     1.03    0.389
Error      40    13.598     0.340
Total      43    14.650
                                   Individual 95% CIs For Mean
                                   Based on Pooled StDev
Level       N      Mean     StDev  --+---------+---------+---------+----
1           7    3.9757    0.8480            (----------*-----------)
2          12    4.1400    0.3579                   (-------*--------)
3           5    3.6800    0.5956   (------------*------------)
4          20    3.8300    0.5829             (------*-----)
                                   --+---------+---------+---------+----
Pooled StDev =   0.5831            3.20      3.60      4.00      4.40
```

One-way ANOVA: PTH_MD versus Quarters

```
Analysis of Variance for PTH_MD
Source     DF        SS        MS        F        P
Quarters    3      72.7      24.2     0.68    0.572
Error      40    1433.6      35.8
Total      43    1506.3
                                   Individual 95% CIs For Mean
                                   Based on Pooled StDev
Level       N      Mean     StDev  -----+---------+---------+---------+-
1           7    10.959     8.365   (----------*-----------)
2          12    14.819     6.947               (--------*--------)
3           5    14.528     4.101          (-------------*-------------)
4          20    14.034     4.677               (------*------)
                                   -----+---------+---------+---------+-
Pooled StDev =    5.987               8.0      12.0      16.0      20.0
```

One-way ANOVA: PTH_SD versus Quarters

```
Analysis of Variance for PTH_SD
Source     DF        SS        MS        F        P
Quarters    3      92.0      30.7     0.61    0.615
Error      40    2024.9      50.6
Total      43    2116.9
                                   Individual 95% CIs For Mean
                                   Based on Pooled StDev
Level       N      Mean     StDev  -+---------+---------+---------+-----
1           7    15.329     9.939   (----------*----------)
2          12    19.834     8.219              (--------*-------)
3           5    18.912     5.441        (------------*------------)
4          20    18.219     5.481             (-----*------)
                                   -+---------+---------+---------+-----
Pooled StDev =    7.115          10.0      15.0      20.0      25.0
```

ET_MD =Equal Temperament Standard, Mean Deviation
ET_SD =Equal Temperament Standard, Standard Deviation
JST_MD=Just Intonation Standard, Mean Deviation
JST_SD =Just Intonation Standard, Standard Deviation
PTH_MD=Pythagorean Standard, Mean Deviation
PTH_SD=Pythagorean Standard, Standard Deviation
sqrt before a variable name indicates a square root transformation

APPENDIX I -- DISCRIMINANT ANALYSIS ON ORIGINAL DATA

Stepwise Discriminant Analysis on Original Data Without Country

Variables Entered/Removed[a,b,c,d]

		Wilks' Lambda							
						Exact F			
Step	Entered	Statistic	df1	df2	df3	Statistic	df1	df2	Sig.
1	ET_MD_SR	.837	1	3	89.000	5.797	3	89.000	.001
2	PT_MD_SR	.707	2	3	89.000	5.547	6	176.000	.000

At each step, the variable that minimizes the overall Wilks' Lambda is entered.

a. Maximum number of steps is 12.

b. Minimum partial F to enter is 3.84.

c. Maximum partial F to remove is 2.71.

d. F level, tolerance, or VIN insufficient for further computation.

Variables in the Analysis

Step		Tolerance	F to Remove	Wilks' Lambda
1	ET_MD_SR	1.000	5.797	
2	ET_MD_SR	.412	10.393	.958
	PT_MD_SR	.412	5.364	.837

Variables Not in the Analysis

Step		Tolerance	Min. Tolerance	F to Enter	Wilks' Lambda
0	ET_MD_SR	1.000	1.000	5.797	.837
	ET_SD_SR	1.000	1.000	5.675	.839
	JS_MD_SR	1.000	1.000	.597	.980
	JS_SD_SR	1.000	1.000	.907	.970
	PT_MD_SR	1.000	1.000	1.308	.958
	PT_SD_SR	1.000	1.000	1.770	.944
1	ET_SD_SR	.011	.011	.480	.823
	JS_MD_SR	.688	.688	1.907	.785
	JS_SD_SR	.880	.880	2.632	.768
	PT_MD_SR	.412	.412	5.364	.707
	PT_SD_SR	.331	.331	4.749	.720
2	ET_SD_SR	.011	.011	.411	.697
	JS_MD_SR	.492	.294	2.892	.643
	JS_SD_SR	.633	.296	3.021	.640
	PT_SD_SR	.034	.034	.265	.701

Wilks' Lambda

Step	Number of Variables	Lambda	df1	df2	df3	Exact F Statistic	df1	df2	Sig.
1	1	.837	1	3	89	5.797	3	89.000	1.149E-03
2	2	.707	2	3	89	5.547	6	176.000	2.738E-05

Standardized Canonical Discriminant Function Coefficients

	Function 1	Function 2
ET_MD_SR	1.554	-.118
PT_MD_SR	-1.116	1.088

Functions at Group Centroids

QUARTERS	Function 1	Function 2
1.00	1.320	-.463
2.00	.673	.352
3.00	-.630	-8.22E-02
4.00	-8.50E-02	1.508E-02

Unstandardized canonical discriminant functions evaluated at group means

Classification Function Coefficients

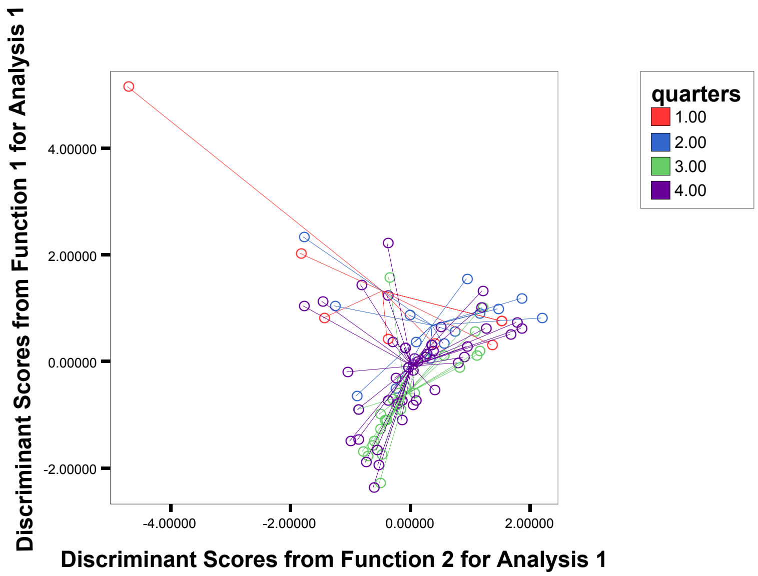
	QUARTERS 1.00	2.00	3.00	4.00
ET_MD_SR	1.409	.148	-2.108	-1.153
PT_MD_SR	3.765	5.777	7.006	6.377
(Constant)	-9.521	-12.522	-10.851	-10.950

Fisher's linear discriminant functions

Classification Results[a]

		QUARTERS	Predicted Group Membership 1.00	2.00	3.00	4.00	Total
Original	Count	1.00	3	4	0	1	8
		2.00	2	10	2	1	15
		3.00	1	4	18	4	27
		4.00	5	10	16	12	43
	%	1.00	37.5	50.0	.0	12.5	100.0
		2.00	13.3	66.7	13.3	6.7	100.0
		3.00	3.7	14.8	66.7	14.8	100.0
		4.00	11.6	23.3	37.2	27.9	100.0

a. 46.2% of original grouped cases correctly classified.



Stepwise Discriminant Analysis on Original Data with Country

Variables Entered/Removed[a,b,c,d]

		Wilks' Lambda											
						Exact F				Approximate F			
Step	Entered	Statistic	df1	df2	df3	Statistic	df1	df2	Sig.	Statistic	df1	df2	Sig.
1	ITALY	.794	1	3	89.000	7.674	3	89.000	.000				
2	GERMANY	.608	2	3	89.000	8.274	6	176.000	.000				
3	SPAIN	.484	3	3	89.000					8.190	9	211.886	.000

At each step, the variable that minimizes the overall Wilks' Lambda is entered.

a. Maximum number of steps is 26.

b. Minimum partial F to enter is 3.84.

c. Maximum partial F to remove is 2.71.

d. F level, tolerance, or VIN insufficient for further computation.

Variables in the Analysis

Step		Tolerance	F to Remove	Wilks' Lambda
1	ITALY	1.000	7.674	
2	ITALY	.814	10.229	.821
	GERMANY	.814	8.972	.794
3	ITALY	.706	15.227	.737
	GERMANY	.732	11.276	.672
	SPAIN	.842	7.485	.608

Variables Not in the Analysis

Step		Tolerance	Min. Tolerance	F to Enter	Wilks' Lambda
0	ET_MD_SR	1.000	1.000	5.797	.837
	ET_SD_SR	1.000	1.000	5.675	.839
	JS_MD_SR	1.000	1.000	.597	.980
	JS_SD_SR	1.000	1.000	.907	.970
	PT_MD_SR	1.000	1.000	1.308	.958
	PT_SD_SR	1.000	1.000	1.770	.944
	ITALY	1.000	1.000	7.674	.794
	GERMANY	1.000	1.000	6.488	.821
	FRANCE	1.000	1.000	1.230	.960
	RUSSIA	1.000	1.000	.380	.987
	ENGLAND	1.000	1.000	3.101	.905
	POLAND	1.000	1.000	6.912	.811
	SPAIN	1.000	1.000	3.354	.898
1	ET_MD_SR	.949	.949	3.452	.711
	ET_SD_SR	.944	.944	3.266	.715
	JS_MD_SR	.956	.956	.888	.771
	JS_SD_SR	.953	.953	1.876	.747
	PT_MD_SR	.858	.858	1.651	.752
	PT_SD_SR	.887	.887	1.370	.759
	GERMANY	.814	.814	8.972	.608
	FRANCE	.984	.984	.898	.771
	RUSSIA	1.000	1.000	.330	.786
	ENGLAND	.999	.999	2.693	.728
	POLAND	.999	.999	6.006	.659
	SPAIN	.936	.936	5.367	.672
2	ET_MD_SR	.884	.758	2.998	.551
	ET_SD_SR	.863	.744	2.974	.552
	JS_MD_SR	.950	.795	.427	.600
	JS_SD_SR	.953	.786	1.732	.574
	PT_MD_SR	.826	.752	1.625	.576
	PT_SD_SR	.857	.768	1.258	.583
	FRANCE	.719	.595	2.216	.565
	RUSSIA	.993	.809	.106	.606
	ENGLAND	.945	.770	.906	.590
	POLAND	.877	.715	2.174	.566
	SPAIN	.842	.706	7.485	.484
3	ET_MD_SR	.880	.680	2.967	.438
	ET_SD_SR	.858	.666	2.980	.438
	JS_MD_SR	.949	.689	.418	.477
	JS_SD_SR	.937	.671	2.215	.449
	PT_MD_SR	.809	.638	1.945	.453
	PT_SD_SR	.847	.658	1.289	.463
	FRANCE	.653	.490	3.607	.430
	RUSSIA	.992	.704	.063	.483
	ENGLAND	.938	.687	.544	.475
	POLAND	.861	.632	1.320	.462

Since all tuning variables were removed, the rest of this test's results are inconsequential to the study.

ET_MD =Equal Temperament Standard, Mean Deviation
ET_SD =Equal Temperament Standard, Standard Deviation
JST_MD=Just Intonation Standard, Mean Deviation
JST_SD =Just Intonation Standard, Standard Deviation
PTH_MD=Pythagorean Standard, Mean Deviation
PTH_SD=Pythagorean Standard, Standard Deviation
SR after a variable name indicates a square root transformation

APPENDIX J -- DISCRIMINANT ANALYSIS ATTEMPT ON AVERAGED DATA

Stepwise Statistics for Average data

Variables Entered/Removed[a,b,c]

At each step, the variable that minimizes the overall Wilks' Lambda is entered.

a. Maximum number of steps is 26.

b. Minimum partial F to enter is 3.84.

c. Maximum partial F to remove is 2.71.

d. F level, tolerance, or VIN insufficient for further computation.

e. No variables are qualified for the analysis.

Variables in the Analysis

Variables Not in the Analysis

Step		Tolerance	Min. Tolerance	F to Enter	Wilks' Lambda
0	ITALY	1.000	1.000	1.597	.893
	GERMANY	1.000	1.000	.479	.965
	FRANCE	1.000	1.000	1.755	.884
	RUSSIA	1.000	1.000	.383	.972
	ENGLAND	1.000	1.000	2.424	.846
	POLAND	1.000	1.000	.383	.972
	SPAIN	1.000	1.000	1.435	.903
	ET_MD	1.000	1.000	.624	.955
	ET_SD	1.000	1.000	.599	.957
	JST_MD	1.000	1.000	.474	.966
	JST_SD	1.000	1.000	.950	.934
	PTH_MD	1.000	1.000	.676	.952
	PTH_SD	1.000	1.000	.606	.957

Wilks' Lambda[a]

a. No variables are qualified for the analysis.

Stepwise Statistics for transformed average data

Variables Entered/Removed[a,b,c]

At each step, the variable that minimizes the overall Wilks' Lambda is entered.

a. Maximum number of steps is 26.

b. Minimum partial F to enter is 3.84.

c. Maximum partial F to remove is 2.71.

d. F level, tolerance, or VIN insufficient for further computation.

e. No variables are qualified for the analysis.

Variables in the Analysis

Variables Not in the Analysis

Step		Tolerance	Min. Tolerance	F to Enter	Wilks' Lambda
0	ET_MD_SR	1.000	1.000	.535	.961
	ET_SD_SR	1.000	1.000	.538	.961
	JS_MD_SR	1.000	1.000	.383	.972
	JS_SD_SR	1.000	1.000	1.032	.928
	PT_MD_SR	1.000	1.000	1.422	.904
	PT_SD_SR	1.000	1.000	1.190	.918
	ITALY	1.000	1.000	1.597	.893
	GERMANY	1.000	1.000	.479	.965
	FRANCE	1.000	1.000	1.755	.884
	RUSSIA	1.000	1.000	.383	.972
	ENGLAND	1.000	1.000	2.424	.846
	POLAND	1.000	1.000	.383	.972
	SPAIN	1.000	1.000	1.435	.903

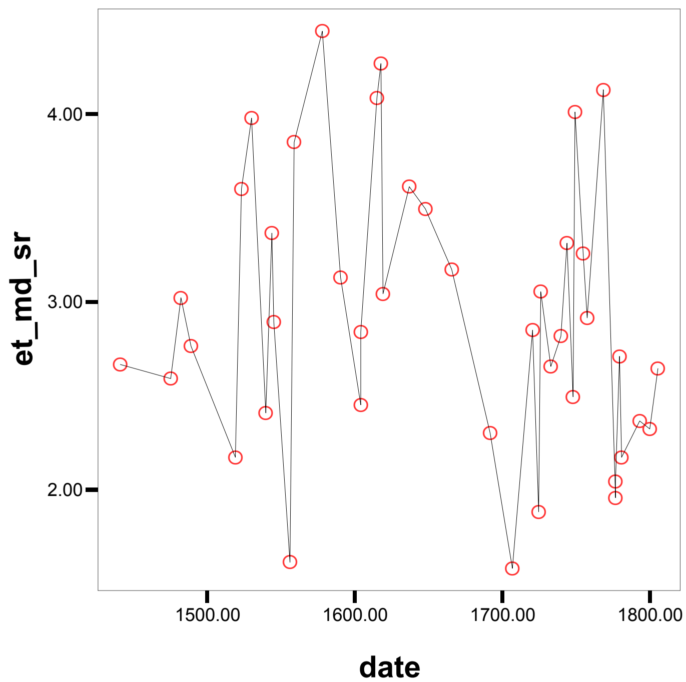
Wilks' Lambda[a]

a. No variables are qualified for the analysis.

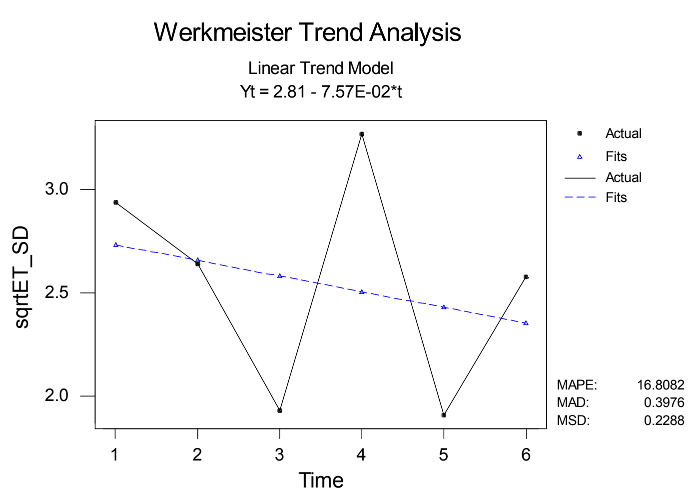
ET_MD =Equal Temperament Standard, Mean Deviation
ET_SD =Equal Temperament Standard, Standard Deviation
JST_MD=Just Intonation Standard, Mean Deviation
JST_SD =Just Intonation Standard, Standard Deviation
PTH_MD=Pythagorean Standard, Mean Deviation
PTH_SD=Pythagorean Standard, Standard Deviation
SR after a variable name indicates a square root transformation

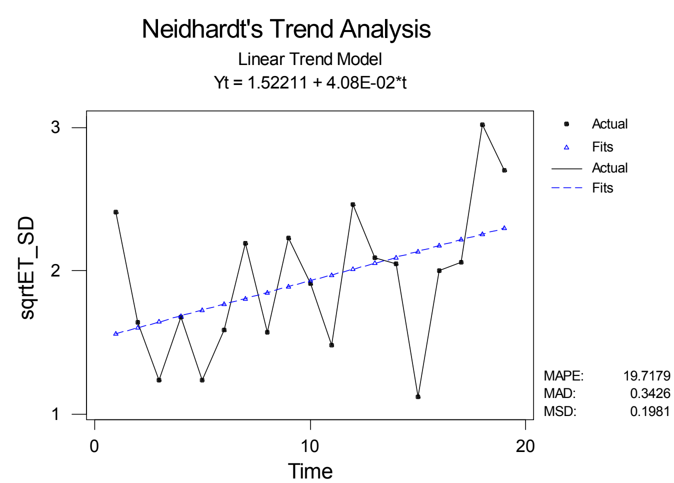
APPENDIX K -- VARIOUS REPRESENTATIONS OF THE DATA

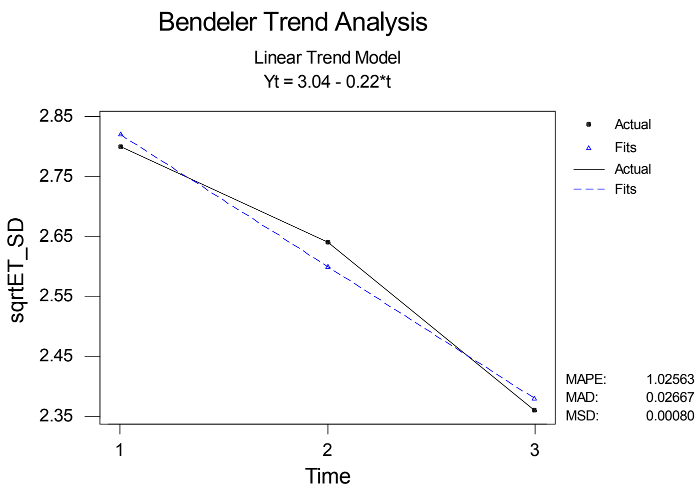
Dot plot of mean deviations from Equal Temperament over time

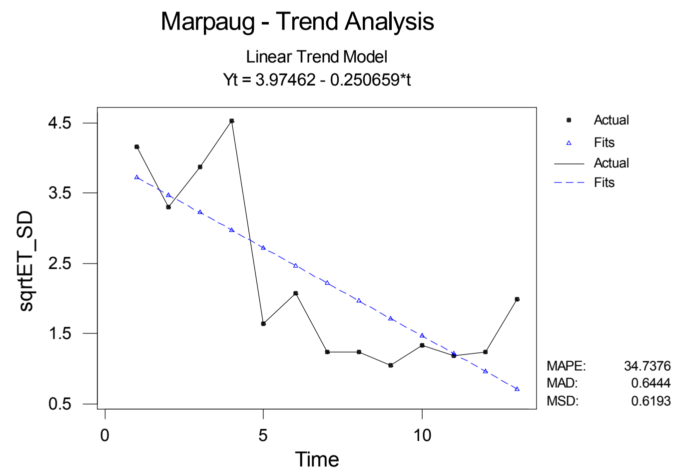


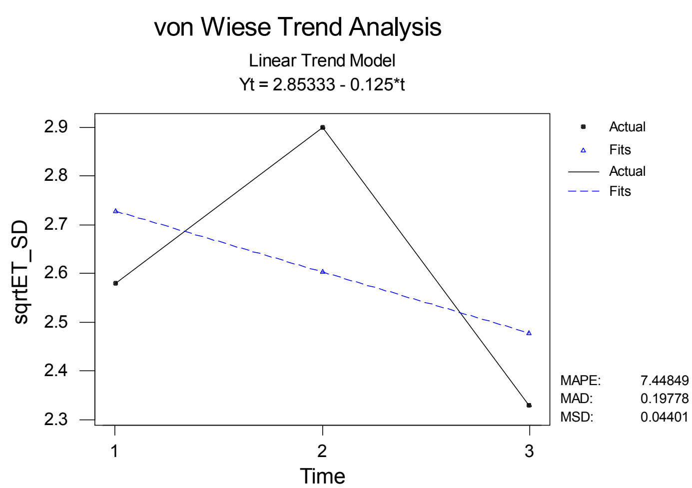
Time series plots of individual Well Temperament advocates with more than two tunings published











ET_MD =Equal Temperament Standard, Mean Deviation
ET_SD =Equal Temperament Standard, Standard Deviation
JST_MD=Just Intonation Standard, Mean Deviation
JST_SD =Just Intonation Standard, Standard Deviation
PTH_MD=Pythagorean Standard, Mean Deviation
PTH_SD=Pythagorean Standard, Standard Deviation
sqr before a variable name indicates a square root transformation

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